

# DISTINGUISHING BETWEEN EXOTIC SYMPLECTIC STRUCTURES

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**ABSTRACT.** We investigate the uniqueness of so-called exotic structures on certain exact symplectic manifolds by looking at how their symplectic properties change under small nonexact deformations of the symplectic form. This allows us to distinguish between two examples based on those found in [16, 17], even though their classical symplectic invariants such as symplectic cohomology vanish. We also exhibit, for any  $n$ , an exact symplectic manifold with  $n$  distinct but exotic symplectic structures, which again cannot be distinguished by symplectic cohomology.

## 1. INTRODUCTION

This paper concerns the uniqueness of exact symplectic structures on Liouville domains, an area which has seen considerable recent development. In many situations, such as those coming from cotangent bundles or affine varieties, a Liouville domain  $M$  carries what is considered to be a “standard” symplectic form. As we shall recap in this introduction, there are now known to be many examples of Liouville domains with exact symplectic forms which are not equivalent to the standard ones. Any such form will be called “exotic” in this paper.

Historically, Gromov [12] was the first to exhibit a nonstandard exact symplectic structure on Euclidean space, although, whereas the standard symplectic structure is Liouville, Gromov’s is not known to be (see Section 2 for the relevant definitions). The first exotic structures on  $\mathbb{R}^{4n}$  (for  $4n \geq 8$ ) known to be Liouville were discovered by Seidel-Smith [29], later extended by McLean [20] to cover all even dimensions greater than 8. McLean actually found infinitely many such pairwise-distinct nonstandard symplectic structures, which were all distinguished by considering their symplectic cohomology  $SH^*(M)$ .

More recently, Fukaya categorical techniques have been used by Maydanskiy-Seidel [17] (refining earlier work of Maydanskiy [16]) to find exotic symplectic structures on  $T^*S^n$  (for  $n \geq 3$ ). These are shown to be nonstandard by proving that they contain no homologically essential exact Lagrangian  $S^n$ , in contrast to the zero-section for the standard symplectic form. Similar results have also been obtained using the work of Bourgeois-Ekholm-Eliashberg [5] again using symplectic/contact cohomology-type invariants. Such results have been further extended by Abouzaid-Seidel [1] to show the existence of infinitely many distinct exotic structures on any affine variety of real dimension  $\geq 6$ , again distinguished using symplectic cohomology.

In this paper, we shall consider six-dimensional symplectic manifolds of the types considered by Maydanskiy [16] and Maydanskiy-Seidel [17]. In [17], infinitely many ways are presented of constructing a nonstandard  $T^*S^3$ , but the question of whether all these constructions actually yield symplectically distinct manifolds is left open. We shall not answer that question, but instead we shall consider what happens when we add a 2-handle to such an exotic  $T^*S^3$ . The result will be diffeomorphic to a manifold constructed in [16], which again contains no exact Lagrangian  $S^3$ .

Specifically, we shall consider the manifolds given by the diagrams in Figure 1.1. The meaning of such diagrams will be explained in Section 2. Briefly, our main method of constructing symplectic manifolds  $E^6$  will be as Lefschetz fibrations over  $\mathbb{C}$ . To run this construction, the input data consists of a symplectic manifold  $M^4$  and an ordered collection of Lagrangian spheres in  $M^4$  (see Lemma 2.2). In Figure 1.1, we can associate to each path some Lagrangian sphere in a 4-dimensional Milnor fibre, which is our required data.

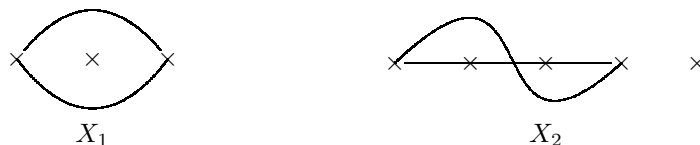


FIGURE 1.1.

Each of these spaces is diffeomorphic to  $T^*S^3 \cup 2\text{-handle}$ . There is a standard way of attaching a 2-handle to  $T^*S^3$  [31] such that we still get an exact Lagrangian sphere inherited from the zero-section. However, neither  $X_1$  nor  $X_2$  contains such a sphere, so are considered exotic. In addition,  $X_1, X_2$  both have vanishing symplectic cohomology. This is not proved in [16, 17] and so we include this calculation in Section 7, and has the consequence (already proven for  $X_1$  in [16]) that  $X_1$  and  $X_2$  actually contain no exact Lagrangian submanifolds (such symplectic manifolds are sometimes called “empty”). Despite the usual collection of invariants being insufficient to distinguish these two manifolds, we shall nevertheless prove

**Theorem 1.1.**  *$X_1$  and  $X_2$  are not symplectomorphic.*

We shall then extend our methods to prove

**Theorem 1.2.** *Pick any  $n \geq 1$ . Then there exists a manifold  $M$  (diffeomorphic to  $T^*S^3$  with  $n$  2-handles attached), which supports exact symplectic forms  $\omega_1, \dots, \omega_{n+1}$  such that, with respect to each  $\omega_i$ ,  $(M, \omega_i)$  is Liouville and contains no exact Lagrangian submanifolds, but such that there exists no diffeomorphism  $\phi$  of  $M$  such that  $\phi^*\omega_i = \omega_j$  for  $i \neq j$ .*

The main technique used in this paper is to consider what happens after a nonexact deformation of the symplectic structure. For any 2-form  $\beta \in H^2(X_i; \mathbb{R})$ , we can consider an arbitrarily small nonexact deformation of  $\omega$  to  $\omega + \epsilon\beta$ . If this new form is still symplectic, we can look at the symplectic properties of these new symplectic manifolds. (In the case of  $X_1$  and  $X_2$  above  $H^2(X_i; \mathbb{R}) = \mathbb{R}$ , so Moser’s argument

tells us that the way we can perform such a deformation is essentially unique, in a sense which will be made precise in Section 6.) We discover that, after an arbitrarily small deformation,  $X_1$  (which with our original exact form contains no Lagrangian  $S^3$ ) does in fact contain such a sphere, an interesting phenomenon in its own right which is explained in Section 3.

In contrast, after such a deformation,  $X_2$  still contains no homologically essential Lagrangian sphere. The proof of this fact requires rerunning the argument of [17], except that somewhat more care needs to be exercised in the use of Floer cohomology groups, owing to the nonexactness of our deformed situation. This is the content of Section 5.

In general, given a symplectic manifold  $M$  (satisfying certain topological assumptions), we can consider the set  $\Gamma_1 \subset \mathbb{P}(H^2(M; \mathbb{R}))$ , of directions in which we get no homologically essential Lagrangian sphere inside  $M$  after an arbitrarily small deformation of the symplectic form in that direction. We show that this is a symplectic invariant, which completes the proof of Theorem 1.1. Finally, in Section 8 these ideas are extended to prove Theorem 1.2.

*Acknowledgements* I would like to thank Paul Seidel whose idea it originally was to distinguish such exotic manifolds by deforming their symplectic structures. I would also like to thank my PhD supervisor Ivan Smith for many helpful discussions, as well as for useful comments on earlier drafts of this work. Thanks also go to Gabriel Paternain, Richard Thomas and the anonymous referee for pointing out several errors in earlier drafts of this paper. During this research I was partially supported by European Research Council grant ERC-2007-StG-205349

## 2. LEFSCHETZ FIBRATIONS

In this section, we recall the standard notions of Picard-Lefschetz theory. The treatment here largely follows that of [28, Part III], but we shall adapt the presentation there to include certain nonexact symplectic manifolds, as we want to consider arguments involving nonexact deformations of our symplectic form.

Let  $(M, \omega)$  be a noncompact symplectic manifold. We say  $(M, \omega)$  is *convex at infinity* if there exists a contact manifold  $(Y, \alpha)$  which splits  $M$  into two parts: a relatively compact set  $M^{in}$ ; and  $M^{out}$ , which is diffeomorphic to the positive symplectization of  $(Y, \alpha)$  where, in a neighbourhood of  $Y$ , we have a 1-form  $\theta$  satisfying  $d\theta = \omega$  and  $\theta|_Y = \alpha$ . Such a contact manifold is canonically identified up to contactomorphism. If  $\theta$  can be defined on the whole of  $M$ , we call  $(M, \theta)$  a *Liouville manifold*.

Given a compact symplectic manifold with boundary  $M$  such that, in a neighbourhood of the boundary, we have a primitive  $\theta$  of the symplectic form which makes the boundary contact, we say  $M$  has *convex boundary*. If  $\theta$  is defined everywhere,  $(M, \theta)$  is usually called a *Liouville domain*. Given a symplectic manifold with convex boundary, we can complete  $M$  canonically to get a symplectic manifold convex at infinity,

$$\widehat{M} = M \cup_{\partial M} [0, \infty) \times \partial M,$$

with forms  $\widehat{\theta} = e^r \theta$  and  $\widehat{\omega} = d\widehat{\theta}$  on the collar, where  $r$  denotes the coordinate on  $[0, \infty)$ .

**2.1. Definition.** Let  $(E, \omega)$  be a compact symplectic manifold with corners such that, near the boundary,  $\omega = d\theta$  for some form  $\theta$  which makes the codimension 1 strata contact, and let  $\pi: E \rightarrow S$  be a proper map to a compact Riemann surface with boundary such that the following conditions hold:

- There exists a finite set  $E^{crit} \subset E$  such that  $D\pi_x$  is a submersion for all  $x \notin E^{crit}$ , and such that  $D^2\pi_x$  is nondegenerate for all  $x \in E^{crit}$ , which means that locally we can find charts such that  $\pi(z) = \sum z_i^2$ . We denote by  $S^{crit}$  the image of  $E^{crit}$  and require that  $S^{crit} \subset S \setminus \partial S$ . We also assume, for sake of notational convenience, that there is at most 1 element of  $E^{crit}$  in each fibre.
- For all  $z \notin S^{crit}$  the fibre  $E_z = \pi^{-1}(z)$  becomes a symplectic manifold with convex boundary with respect to  $\omega|_{E_z}$ . This means that we get a splitting of tangent spaces

$$TE_x = TE_x^h \oplus TE_x^v,$$

where the vertical part  $TE_x^v$  is the kernel  $\ker(D\pi_x)$  and the horizontal part  $TE_x^h$  is the orthogonal complement of  $TE_x^v$  with respect to  $\omega$ .

- At every point  $x \in E$  such that  $z = \pi(x) \in \partial S$ , we have  $TS = T(\partial S) + D\pi(TE_x)$ . This implies that  $\pi^{-1}(\partial S)$  is a boundary stratum of  $E$  of codimension 1, which we shall call the *vertical boundary*, denoted  $\partial^v E$ . The union of boundary faces of  $E$  not contained in  $\partial^v E$  we shall call the *horizontal boundary* of  $E$ , denoted  $\partial^h E$ .
- If  $F$  is a boundary face of  $E$  not contained in  $\partial^v E$ , then  $\pi|_F: F \rightarrow S$  is a smooth fibration, which implies that any fibre is smooth near its boundary. We also want the horizontal boundary  $\partial^h E$  to be horizontal with respect to the above splitting, so that parallel transport (see below) will be well-defined along the boundary.

**Definition 2.1.** *If all the above holds we call  $(E, \pi, \omega)$  a compact convex Lefschetz fibration. For ease of notation, in what follows we shall often call  $(E, \pi, \omega)$  simply a Lefschetz fibration, suppressing the extra adjectives.*

The splitting of tangent spaces into horizontal and vertical subspaces means that we have a connection over  $S \setminus S^{crit}$ , and hence *symplectic parallel transport* maps. In other words, for a path  $\gamma: [0, 1] \rightarrow S$  which misses  $S^{crit}$ , our connection defines a symplectomorphism  $\phi_\gamma: E_{\gamma(0)} \rightarrow E_{\gamma(1)}$ .

There is a method [20] of completing  $E$  to a symplectic manifold  $\widehat{E}$  which is convex at infinity, such that we get a map  $\widehat{\pi}: \widehat{E} \rightarrow \widehat{S}$  to the completion of the base. When  $S$  is a disc  $\mathbb{D}$ , this is done as follows: firstly, the horizontal boundary  $\partial^h E$  is just  $\partial M \times \mathbb{D}$ , where  $M$  is a smooth fibre, and we can attach  $\partial M \times [0, \infty) \times \mathbb{D}$  to  $\partial^h E$  in the same as we complete a symplectic manifold with convex boundary. This gives us a new manifold we shall call  $E_1$  and we can extend  $\pi$  to  $\pi_1$  on  $E_1$  in the obvious way. Now consider  $\pi_1^{-1}(\partial \mathbb{D}) = N$ . Attach to this  $N \times [0, \infty)$  and call the resulting manifold  $\widehat{E}$ , over which we can extend  $\pi_1$  to  $\widehat{\pi}$ . More details can be found in [20,

Section 2]. This map  $\widehat{\pi}$  restricts to  $\pi$  on the subsets corresponding to  $E$  and  $\mathbb{D}$  and outside we have a local model looking like the completion of the mapping cone for some symplectic map  $\mu$  which we shall call the *outer monodromy* of the Lefschetz fibration  $E$ . Given this, we shall also talk in this paper about Lefschetz fibrations over  $\mathbb{C}$ , which are understood to be the completions of Lefschetz fibrations over some disc  $\mathbb{D}_R \subset \mathbb{C}$ , in the sense of Definition 2.1.

**2.2. Vanishing cycles.** We can use the parallel transport maps to introduce the notion of a *vanishing cycle*. Choose an embedded path  $\gamma: [0, 1] \rightarrow S$  such that  $\gamma^{-1}(S^{crit}) = \{1\}$ . We can consider the set of points which tend to the critical point  $y = \gamma(1)$  under our parallel transport maps

$$V_\gamma = \left\{ x \in E_{\gamma(0)} : \lim_{t \rightarrow 1} \phi_{\gamma|_{[0,t]}}(x) = y \right\}.$$

This set  $V_\gamma$  is called the *vanishing cycle* associated to the *vanishing path*  $\gamma$ . The vanishing cycle is actually a Lagrangian sphere in the fibre [27] and if we take the *Lefschetz thimble*, the union of the images of the vanishing cycle as we move along  $\gamma$  together with the critical point, we get a Lagrangian ball  $\Delta_\gamma$  in the total space  $E$ . In fact,  $\Delta_\gamma$  is the unique embedded Lagrangian ball that lies over  $\gamma$ . These vanishing cycles come together with the extra datum of a “framing” [27, Lemma 1.14], meaning a parametrization  $f: S^n \rightarrow V$ . Here, two framings  $f_1, f_2$  are equivalent if  $f_2^{-1}f_1$  can be deformed inside  $Diff(S^n)$  to an element of  $O(n+1)$ , but this framing information is irrelevant in the dimensions in which we work, so shall neglect to mention framings in what follows.

**2.3. Constructing Lefschetz fibrations.** Given a Lefschetz fibration  $(E, \pi)$ , we can pick a smooth reference fibre  $E_z$  and a collection of vanishing paths  $\gamma_i$ , one for each critical point, which all finish at  $z$ , but which are otherwise disjoint. This then gives us a symplectic manifold  $M = E_z$  and a collection of vanishing cycles  $V_i \subset M$  associated to the  $\gamma_i$ . For our purposes, in constructing symplectic manifolds, it is important to note that we can go the other way as in the following lemma, taken from [28, Lemma 16.9] but with unnecessary assumptions of exactness removed.

**Lemma 2.2.** *Suppose we have a collection  $(V_1, \dots, V_m)$  of (framed) Lagrangian spheres in a symplectic manifold  $M$  with convex boundary. On the disc  $\mathbb{D}$ , choose a base point  $z$ , and a distinguished basis of vanishing paths  $\gamma_1, \dots, \gamma_m$  all of which have one endpoint at  $z$ . Then there is a compact convex Lefschetz fibration  $\pi: E \rightarrow \mathbb{D}$ , whose critical values are precisely the endpoints  $\gamma_1(1), \dots, \gamma_m(1)$ ; this comes with an identification  $E_z = M$ , under which the (framed) vanishing cycles  $V_{\gamma_k}$  correspond to  $V_k$ .*

This will be the technique used to construct the symplectic manifolds considered in this paper. However, in order to do this, we need to identify a collection of Lagrangian spheres in a given symplectic manifold  $M$ . In the case where  $M$  itself admits a Lefschetz fibration, we shall do this by considering matching cycles.

**2.4. Matching cycles.** Consider a Lefschetz fibration  $\pi: M \rightarrow S$  and an embedded path  $\gamma: [0, 1] \rightarrow S$  such that  $\gamma^{-1}(S^{crit}) = \{0, 1\}$ . In the fibre  $\pi^{-1}(\gamma(\frac{1}{2}))$ , we get two vanishing cycles, one coming from either endpoint. If they agree, then

parallel transport allows us to glue the two thimbles together to obtain a smooth Lagrangian sphere  $V \subset M$ . We shall call  $\gamma$  a *matching path*, and  $V$  the associated *matching cycle*.

In this paper we shall usually work in situations where the vanishing cycles do agree exactly so that we do get matching cycles, but occasionally we will have the situation where the two vanishing cycles are not equal, but are merely Hamiltonian isotopic. In this situation we may appeal to the following result of [4, Lemma 8.4]:

**Lemma 2.3.** *Let  $(M, \omega)$  be a symplectic manifold with a Lefschetz fibration  $\pi: M \rightarrow \mathbb{C}$  and let  $\gamma: [0, 1] \rightarrow \mathbb{C}$  be a path such that  $\gamma^{-1}(S^{\text{crit}}) = \{0, 1\}$ . Suppose that the two vanishing cycles  $V_0, V_1 \subset M_{\gamma(\frac{1}{2})}$  coming from either end of this path are Hamiltonian isotopic for some compactly supported Hamiltonian  $H_s$  defined on the fibre  $M_{\gamma(\frac{1}{2})}$ . Then  $M$  contains a Lagrangian sphere.*

Matching cycles will be used for our main method of construction. We take a symplectic manifold  $(M, \omega)$  equipped with a Lefschetz fibration and consider an ordered collection of matching paths. In favourable circumstances these will give rise to a family of framed Lagrangian spheres  $(V_1, \dots, V_n) \subset M$  and we now apply Lemma 2.2 to construct a new Lefschetz fibration  $(E, \pi)$ .

**2.5. Maydanskiy's examples.** Figure 2.1 shows the examples considered in [16]. Although higher-dimensional examples are also considered in [16], the meaning of all such diagrams in this paper is that we take the symplectic manifold  $M^4$  built according to Lemma 2.2 by taking fibre  $T^*S^1$  and vanishing cycles given by the zero-section, one for each cross. The lines in Figure 2.1 are then matching paths which yield the spheres required to apply Lemma 2.2 again to obtain  $E^6$ . The fact that the paths in Figure 2.1 actually do give matching cycles will for us be a consequence of the method of construction considered in the next section.

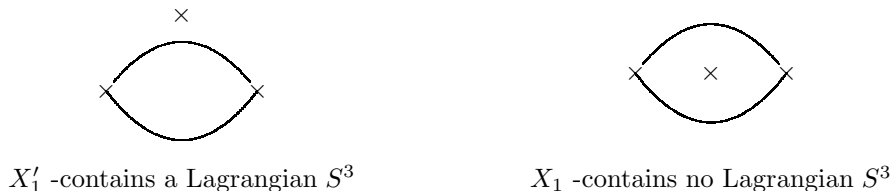


FIGURE 2.1.

Maydanskiy [16] proves that the two symplectic manifolds in Figure 2.1 are diffeomorphic (they are both  $T^*S^3 \cup 2\text{-handle}$ ) but are not symplectomorphic.  $X'_1$  is just  $T^*S^3$  with a Weinstein 2-handle attached as in [31] and contains an exact Lagrangian sphere inherited from the zero-section of  $T^*S^3$ . In contrast,  $X_1$  contains no exact Lagrangian submanifolds, and so is considered exotic.

One way of thinking about this intuitively is that the manifolds are diffeomorphic because one can construct a smooth isotopy taking the top matching cycle in  $X_1$  and moving it over the critical point in the middle to get  $X'_1$ . The reason this fails to work symplectically is that we are free to move our cycles only by Hamiltonian

isotopies, and we will not then be able to avoid the central critical point (since we cannot displace the zero-section of  $T^*S^1$ ), although the actual proof in [16] has to make use of more sophisticated Floer-theoretic arguments.

### 3. DEFORMATIONS OF SYMPLECTIC STRUCTURES

**Definition 3.1.** *Let  $(E, \omega)$  be a symplectic manifold. By a deformation of the symplectic structure  $(E, \omega)$  we shall mean a smooth 2-form  $\Omega$  on  $\tilde{E} = E \times [0, 1]$  such that*

- $\Omega|_t$  is symplectic on each  $E \times \{t\}$
- $\Omega|_0 = \omega$
- $\iota_v \Omega = 0$  for any  $v \in \ker(D\rho)$  where  $\rho$  is the projection  $\tilde{E} \rightarrow E$ .

*This is equivalent to a smooth 1-parameter family of symplectic forms  $\{\omega_t : t \in [0, 1]\}$  on  $E$  such that  $\omega_0 = \omega$ . We shall denote by  $(\tilde{E}^t, \omega_t)$  the symplectic manifold  $(E \times \{t\}, \Omega|_t)$ .*

We shall consider  $X_1$ , the exotic example of Maydanskiy from the previous section. In this section, we shall prove

**Theorem 3.2.** *There is a deformation  $\tilde{X}_1$  of  $X_1$  such that, for all  $t > 0$ ,  $\tilde{X}_1^t$  contains a Lagrangian sphere.*

**3.1. Constructing a deformation of  $X_1$ .** The fibres of Maydanskiy's examples are  $A_2$  Milnor fibres. For our purposes, which crucially rely on matching paths defining genuine matching cycles without having to rely on Lemma 2.3, we shall work with the specific model as below.

Let  $M$  be the affine variety defined by

$$M = \{z_1^2 + z_2^2 = (z_3 - 1)(z_3 - 2)(z_3 - 3)\} \subset \mathbb{C}^3$$

equipped with symplectic form  $\omega$ , which is the restriction of the standard symplectic form on  $\mathbb{C}^3$ . We may restrict to some compact set  $M^{in} \subset M$  ( $M^{in} \subset B_R \subset \mathbb{C}$  for some sufficiently large  $R$ ), such that  $M^{in}$  is a Liouville domain which becomes a Lefschetz fibration in the sense of Definition 2.1 once we project onto the  $z_3$ -coordinate [28, Section 19b]. It has three critical values, at 1, 2 and 3.

There is a homologically essential Lagrangian sphere  $A$  living over the straightline path joining the two critical points at 1 and 2, which is given by the part of the real locus  $M_{\mathbb{R}}$  living over this path. This sphere is precisely the matching cycle associated to that line. We can do the same with the part of  $M \cap \mathbb{R}\langle x_3, y_1, y_2 \rangle$  living over the interval  $[2, 3]$  to find another Lagrangian sphere  $B$  and we shall take  $A$  and  $B$  to define our standard basis of  $H_2(M; \mathbb{R}) = \mathbb{R}^2$ .

The manifold  $M$  carries an  $S^1$ -action given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and the symplectic form  $\omega$  is invariant under this action.

Every smooth fibre is of the form  $z_1^2 + z_2^2 = \lambda$  for some nonzero  $\lambda = se^{i\alpha}$  and we observe that such a fibre is preserved by the  $S^1$ -action, which in particular means that the parallel transport map associated to a path  $\gamma$  is  $S^1$ -equivariant. This fibre is symplectomorphic to  $T^*S^1$ , where the model we use for  $T^*S^1$  is

$$T^*S^1 = \{(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|q\| = 1, \langle q, p \rangle = 0\}.$$

The symplectomorphism is defined as follows: let  $\hat{z} = ze^{-i\alpha/2}$  and map

$$z \mapsto \left( \frac{\Re(\hat{z})}{\|\Re(\hat{z})\|}, -\Im(\hat{z})\|\Re(\hat{z})\| \right).$$

Note that, for each fibre, the  $S^1$ -orbits are mapped to level sets  $\|p\| = \text{constant}$  so, given that the parallel transport maps are  $S^1$ -equivariant, the vanishing cycle associated to any vanishing path will itself correspond to such a level set.

We shall deform the symplectic structure by introducing 2-forms which are intended to resemble area forms supported near the equators of  $A$  and  $B$ . We therefore consider the 2-form on  $\mathbb{C}^3 \setminus i\mathbb{R}^3$ ,

$$\eta = g_\epsilon \left( \frac{x}{\|x\|} \right) \left( \frac{x_1}{\|x\|^3} dx_2 \wedge dx_3 + \frac{x_2}{\|x\|^3} dx_3 \wedge dx_1 + \frac{x_3}{\|x\|^3} dx_1 \wedge dx_2 \right)$$

where  $g_\epsilon(x) = g_\epsilon(x_3)$  denotes a cutoff function for the  $x_3$ -coordinate which has  $\text{supp}(g_\epsilon) \subset \{|x_3| < \epsilon\}$ .

As  $\eta$  is defined using only coordinates on the real slice  $\mathbb{R}^3 \setminus \{0\}$  and annihilates the radial direction, this is a closed form on  $\mathbb{C}^3 \setminus i\mathbb{R}^3$ . We shall choose  $\epsilon$  such that  $\epsilon < \frac{1}{8R}$ , and apply a translation  $x \mapsto x + (0, 0, 3/2)$ . It is easy to show that  $\eta$  is now well-defined on  $M$ , so that in the Lefschetz fibration  $M^{in} \rightarrow \mathbb{D}_R$ ,  $\eta$  is a closed,  $S^1$ -equivariant 2-form supported in the region lying over  $\{|x_3 - 3/2| < 1/4\}$  and the sphere  $A$  has some nonzero area with respect to  $\eta$ .

Moreover, we can rescale  $\eta$  so that  $\omega + \eta$  is still symplectic on  $M^{in}$ , since the property of being symplectic is an open condition and  $M^{in}$  is compact. Also, since  $M$  is an  $A_2$  Milnor fibre, its boundary  $\partial M$  is topologically the quotient of  $S^3$  by a  $\mathbb{Z}/3$  action and therefore  $H^2(\partial M; \mathbb{R}) = 0$ . This means that, perhaps after rescaling  $\eta$  again,  $M^{in}$  will still have contact boundary.

We repeat the above procedure to obtain another closed 2-form  $\eta'$  on  $M^{in}$ , defined now using the coordinates  $y_1, y_2, x_3$  which is again  $S^1$ -equivariant and is supported over  $\{|x_3 - 5/2| < 1/4\}$  and has the property that

$$\eta(A) = -\eta'(B).$$

We denote by  $\omega_t$  the 2-forms  $\omega + t(\eta + \eta')$  for  $t \in [0, 1]$ , all of which make  $M^{in}$  symplectic with convex boundary.

**Remark 3.3.** *Such a construction can be generalized: choose a finite collection of distinct points  $p_1, \dots, p_{n+1} \in \mathbb{R}$  and consider the affine variety*

$$M_{\mathbf{p}} = \left\{ z_1^2 + z_2^2 = \prod_i (z_3 - p_i) \right\} \subset \mathbb{C}^3,$$

which will be diffeomorphic to the  $A_n$  Milnor fibre, with a basis of  $H_2(M_{\mathbf{p}})$  given by the spheres  $A_i$  living over the straightline path joining  $p_i$  and  $p_{i+1}$ . We may construct a deformation of the symplectic structure on  $M_{\mathbf{p}}$  by adding on 2-forms which are supported on strips lying between the critical points as above.

**3.2. Obstructions to forming matching cycles are purely homological.** We now consider the path  $\gamma_0$  in Figure 3.1, going from 1 to 3 in  $\mathbb{C}$ . We would like this to define a genuine matching cycle, with respect to the parallel transport maps coming from  $\omega_t = \omega + t(\eta + \eta')$  for  $t \in [0, 1]$ . However, we may no longer get a genuine Lefschetz fibration in the sense of Section 2, since the horizontal boundary may no longer be horizontal with respect to our splitting. This means that parallel transport cannot be done near  $\partial^h M$ , but we shall not need this: our vanishing cycles stay within a region away from the boundary, since deforming the symplectic form will only change the parallel transport maps by a small amount.

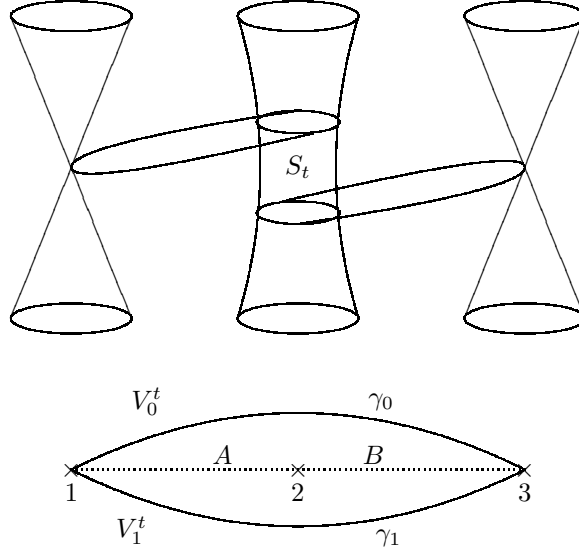


FIGURE 3.1.

Therefore, for any given  $t$ , the path  $\gamma_0$  gives us two circles in the central fibre which we know correspond to level sets  $\|p\| = \text{constant}$ . (In Figure 3.1, the fibres shown at the top are those living over the path  $\gamma_0$ .) These two circles enclose some chain  $S_t$  in the fibre over  $\gamma_0(\frac{1}{2})$ , and the sum of this chain and the two thimbles is homologous to  $[A] + [B]$ , so therefore has symplectic area 0 with respect to  $\omega_t$ . Since the vanishing thimbles are Lagrangian, this means that the chain  $S_t \subset T^*S^1$  must also have zero symplectic area, and therefore  $S_t$  must in fact be empty. In other words, we get a genuine matching cycle for all  $t$ , which we denote  $V_0^t$ . We can do likewise for the path  $\gamma_1$  to obtain the matching cycle  $V_1^t$ .

By the same argument, for any  $t > 0$  we can take a straightline path given by the interval  $[1, 3]$ , which goes over the central critical point at 2, and say that this too will define a matching cycle: in the central nonsmooth fibre we shall either

get, by  $S^1$ -symmetry, the critical point or some circle. However, if we obtained the critical point, then we would have found a Lagrangian in a homology class of positive symplectic area. Which smooth component this circle lives in depends on whether we choose to give the class  $A$  positive or negative area.

Therefore, for  $t > 0$ , we can take a smooth family of paths interpolating between the two matching paths and get a smooth family  $(V_s^t)_{s \in [0,1]}$  of Lagrangian  $S^2$ s joining the two matching cycles. This has the following standard consequence.

**Lemma 3.4.** *For  $t > 0$ ,  $V_0^t$  and  $V_1^t$  are Hamiltonian isotopic.*

*Proof.* We can identify some neighbourhood of  $V_0^t$  with  $T^*S^2$  and, for  $0 \leq s \leq s_0$  for some small  $s_0$ ,  $V_s^t$  will correspond to the graph of some 1-form  $\alpha_s$ . Since  $V_s^t$  is Lagrangian,  $d\alpha_s = 0$ , and therefore  $\alpha_s = df_s$  since  $H^2(V_s^t; \mathbb{R}) = 0$ . We can moreover choose these  $f_s$  smoothly. A direct calculation shows that  $H(x, s) = \frac{d}{ds}(f_s(\rho(x)))$  is a Hamiltonian yielding an isotopy between  $V_0^t$  and  $V_{s_0}^t$ , where here  $\rho: T^*S^2 \rightarrow S^2$  is the standard projection map. We can patch together such isotopies to get from  $V_0^t$  to  $V_1^t$ , and then apply some cutoff function to make our Hamiltonian to be compactly supported.  $\square$

**3.3.  $X_1$  contains a Lagrangian sphere after deformation.** We are now in a position to prove Theorem 3.2. To do this, we shall establish a deformation version of Lemma 2.2. This is stated below in the case where there is just one vanishing cycle, since the general case follows from gluing together such examples.

Suppose we have  $\tilde{M}$ , a deformation of the symplectic structure  $(M, \omega)$ , such that  $\tilde{M}^t$  has convex boundary for all  $t$ , and suppose that we also have  $\tilde{V} \subset \tilde{M}$ , which is the image of an embedding of  $S^n \times [0, 1]$  such that, for all  $t$ , we get a Lagrangian sphere  $\tilde{V}^t \subset (\tilde{M}^t, \omega_t)$ .

Then, by Lemma 2.2, we can construct a Lefschetz fibration  $E^t \rightarrow \mathbb{D}$  from  $\tilde{M}^t$  and  $\tilde{V}^t$  for each  $t$ . We want the family  $E^t$  to comprise a deformation of  $(E^0, \omega_E)$ .

**Proposition 3.5.** *In the above situation, we can construct a bundle of symplectic manifolds  $\tilde{E} \rightarrow [0, 1]$ , such that each fibre  $\tilde{E}^t$  has convex boundary and comes with an identification  $\tilde{E}_z^t \cong \tilde{M}^t$ , under which the vanishing cycle  $V_\gamma$  corresponds to  $\tilde{V}^t$ . After applying a trivialization of this bundle which is the identity over 0, this is a deformation of  $(E^0, \omega_E)$ .*

*Proof.* We closely follow [27, Proposition 1.11]. First we need a neighbourhood theorem, whose proof follows the same argument as that of the standard Lagrangian neighbourhood theorem [18].

**Lemma 3.6.** *Let  $(\tilde{M}, \Omega)$  be a deformation of  $(M, \omega)$ . Suppose we have  $\tilde{V} \subset \tilde{M}$  an embedding of  $V \times [0, 1]$  such that, for all  $t$ , we get a Lagrangian  $\tilde{V}^t \subset (\tilde{M}^t, \omega_t)$ . Then there exists a neighbourhood  $\mathcal{N} \subset T^*V \times [0, 1]$  of the zero-section  $V \times [0, 1]$  and a neighbourhood  $\mathcal{U} \subset \tilde{M}$  of  $\tilde{V}$  and a diffeomorphism  $\phi: \mathcal{N} \rightarrow \mathcal{U}$  such that  $\phi^*\Omega = \beta$  where  $\beta$  is the 2-form on  $T^*V \times [0, 1]$  given by the pullback of the standard symplectic form on  $T^*V$ .*

In our case, we may assume our neighbourhood  $\mathcal{N}$  in Lemma 3.6 is of the form  $\mathcal{N} = T_{\leq \lambda}^* S^n \times [0, 1]$  for some  $\lambda > 0$ , where  $T_{\leq \lambda}^* S^n$  denotes the disc cotangent bundle with respect to the standard metric on  $T^* S^n$ . Given this, we follow [27, Proposition 1.11] which starts by considering the local Lefschetz model  $q: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ ,  $q(z) = \sum z_i^2$ . We also consider the function  $h(z) = \|z\|^4 - |q(z)|^2$ .

When we restrict to  $W \subset \mathbb{C}^{n+1}$  cut out by the inequalities  $h(x) \leq 4\lambda^2$  and  $|q(z)| \leq 1$ , we get a compact Lefschetz fibration  $\pi_W: W \rightarrow \mathbb{D}$ . As explained in [27],  $W$  comes together with an identification  $\psi: \pi_W^{-1}(1) \rightarrow T_{\leq \lambda}^* S^n$ , a neighbourhood  $Y \subset W$  of  $\partial_h W$ , a neighbourhood  $Z$  of  $\partial(T_{\leq \lambda}^* S^n)$  in  $T_{\leq \lambda}^* S^n$  and a diffeomorphism  $\Psi: Y \rightarrow \mathbb{D} \times Z$  which fibres over  $\mathbb{D}$  and agrees with  $\psi$  on  $Y \cap \pi_W^{-1}(1)$ . Let  $\tilde{W} = W \times [0, 1]$  and, by taking the product with  $[0, 1]$ , consider the corresponding  $\tilde{Y}, \tilde{Z}, \tilde{\psi}, \tilde{\Psi}$ .

Now define  $\tilde{M}_-$  to be  $\tilde{M} \setminus (\phi(\mathcal{N} \setminus \tilde{Z}))$  and consider

$$\tilde{E} = \mathbb{D} \times \tilde{M}_- \cup_{\sim} \tilde{W},$$

where the identification made identifies  $\tilde{Y}$  with  $\mathbb{D} \times \phi(\tilde{Z})$  through  $(id \times \phi) \circ \tilde{\Psi}$ . This now has all the required properties.  $\square$

of Theorem 3.2. Using Proposition 3.5, we can construct a deformation  $\tilde{X}_1$  of Maydanskiy's exotic example  $X_1$  and we want to say that we have a Lagrangian sphere  $L^t \subset \tilde{X}_1^t$  for all  $t > 0$ .  $\tilde{X}_1^t$  admits a Lefschetz fibration with two critical points. We take a path joining the two critical points in the Lefschetz fibration on  $X_1$ . If we choose the vanishing paths  $\gamma$  in Proposition 3.5 such that they join together smoothly, then the concatenation of these paths is smooth and yields two vanishing cycles in the central fibre, which are precisely just  $V_0^t$  and  $V_1^t$  from Lemma 3.4, which we know are Hamiltonian isotopic for all  $t > 0$ . We then just apply Lemma 2.3 to find a Lagrangian sphere.  $\square$

**Remark 3.7.** As  $t \rightarrow 0$ , the Lagrangian spheres  $L^t$  degenerate to some singular Lagrangian cycle, which is worse than immersed. In fact, topologically it looks like  $S^3$  with some  $S^1$  in it collapsed to a point. Presumably, pseudoholomorphic curve theory with respect to this cycle is very badly behaved, so that a Floer theory along the lines of [3] cannot be made to work here, although see [14] for some analysis of holomorphic discs on certain similar special Lagrangian cones.

#### 4. FLOER COHOMOLOGY

To consider  $X_2$  and adapt the arguments presented in [17], we shall need to consider the Lagrangian Floer cohomology  $HF(L_0, L_1)$  of two transversely intersecting Lagrangian submanifolds in some symplectic manifold  $(M, \omega)$ . To define this, one has to pick a generic family of almost complex structures  $\mathbf{J} = (J_t)$ , which are usually required to be compatible with  $\omega$ , in the sense that  $g_t(u, v) = \omega(u, J_t v)$  defines a Riemannian metric. However, we shall want to consider  $J_t$  which are  $\omega$ -tame except on a small neighbourhood of  $L_0 \cap L_1$ , where here  $J_t$  is still  $\omega$ -compatible. ( $\omega$ -tame means that  $\omega(u, J_t u) > 0$  for all nonzero  $u$ .) We shall show that, given any such family of almost complex structures  $\mathbf{J} = (J_t)$ , there exists  $\tilde{\mathbf{J}} = (\tilde{J}_t)$  arbitrarily close to it, with the same properties, such that  $HF(L_0, L_1)$  can be defined with respect

to  $(\tilde{J}_t)$ . The key point is that we are using Cauchy-Riemann type operators with totally real boundary conditions, so all the relevant elliptic regularity theory can still be applied.

**Remark 4.1.** *The content of this section, that we can relax the condition on the almost complex structures to define Floer cohomology is probably already known to experts, but we are unaware of any written account of this in the literature.*

**4.1. Setup.** Let  $(M, \omega)$  be a symplectic manifold of dimension  $2n$  and let  $L_0, L_1$  be two Lagrangian submanifolds which intersect transversely. For each intersection  $x$ , fix some small open set  $U_x$  around  $x$  such that  $L_0 \cap L_1 \cap U_x = \{x\}$ . Assume moreover that the  $U_x$  are disjoint. Pick some family  $\mathbf{J} = (J_t)$  of smooth almost complex structures which tame  $\omega$  (this in particular implies that the  $L_k$  are totally real), and which are  $\omega$ -compatible on each  $U_x$ .

We note here for future reference the following lemma due to Frauenfelder [9].

**Lemma 4.2.** *Let  $(M^{2n}, J)$  be an almost complex manifold and  $L^n \subset M$  a totally real submanifold. Then there exists a Riemannian metric  $g$  on  $M$  such that*

- $g(J(p)v, J(p)w) = g(v, w)$  for  $p \in M$  and  $v, w \in T_p M$ ,
- $J(p)T_p L$  is the orthogonal complement of  $T_p L$  for every  $p \in L$ ,
- $L$  is totally geodesic with respect to  $g$ .

Let  $\Sigma$  denote the holomorphic strip  $\mathbb{R} \times [0, 1] \subset \mathbb{C}$ . Given a map  $u: \Sigma \rightarrow M$ , we can consider the  $\bar{\partial}_{\mathbf{J}}$  operator defined by

$$\bar{\partial}_{\mathbf{J}} u(s, t) = \partial_s u(s, t) + J_t(s, t) \partial_t u(s, t).$$

We care about holomorphic maps, which are just those such that  $\bar{\partial}_{\mathbf{J}} u = 0$  and we define the energy of any map  $u$  to be  $E(u) = \int \|\partial_s u\|^2$ .

Let  $\mathcal{M}_{\mathbf{J}}$  denote the set of holomorphic  $u$  as above which also satisfy the boundary conditions  $u(s, 0) \in L_0$ ,  $u(s, 1) \in L_1$  as well as  $E(u) < \infty$ . It is proved in [25] that any such map must have the property that

$$\lim_{s \rightarrow \pm\infty} u(s, t) = x^{\pm},$$

where  $x^{\pm}$  are intersection points in  $L_0 \cap L_1$ . Moreover, the convergence near the ends is exponential in a suitable sense about which we shall say more later. We define  $\mathcal{M}_{\mathbf{J}}(x, y)$  to be the space of finite-energy trajectories as above which converge to  $x$  and  $y$  at the ends.

We want to examine the properties of  $\mathcal{M}_{\mathbf{J}}(x, y)$  and, in particular, determine when it is a smooth manifold, so we follow the standard procedure of Floer [8], in exhibiting  $\mathcal{M}_{\mathbf{J}}(x, y)$  as the zero set of some Fredholm section of a Banach bundle. Much of what follows is already contained in Floer's original work [8], but we shall recall the main details for the reader's convenience.

**4.2. Banach manifolds.** Let  $kp > 2$ . We can consider the Sobolev space  $L_{k;loc}^p(\Sigma, M)$  and define

$$\mathcal{P}_k^p = \left\{ u \in L_{k;loc}^p(\Sigma, M) : u(s, 0) \in L_0, u(s, 1) \in L_1 \right\}.$$

Let  $\Sigma_\rho = \{z \in \Sigma : |\Re z| < \rho\}$ . The topology on  $\mathcal{P}_k^p$  is defined using the basis of open sets given by

$$\mathcal{O}_{u,\rho,\epsilon} = \{v \in \mathcal{P}_k^p : v = \exp_u \xi \text{ on } \Sigma_\rho \text{ and } \|\xi\|_{k,p} < \epsilon \text{ for } p < \rho\}.$$

Here  $u \in \mathcal{P}_k^p$  and  $\rho, \epsilon > 0$ .

For our present purposes, and in order to ensure that we do in fact get a Banach manifold, we shall need to restrict to a subset of  $\mathcal{P}_k^p$  with nice behaviour near intersection points  $x \in L_0 \cap L_1$ . Consider

$$\mathcal{P}_k^p(\cdot, x) = \left\{ u \in \mathcal{P}_k^p : \exists \rho > 0, \exists \xi \in L_{k;loc}^p(\Sigma, T_x M), u(s, t) = \exp_x \xi(s, t) \forall s > \rho \right\}.$$

In other words, we restrict attention to maps which, at one end, look like the exponentiation of some vector field. We impose a similar condition at the other end to define  $\mathcal{P}_k^p(x, \cdot)$ , and then consider  $\mathcal{P}_k^p(x, y)$ .

For  $u \in \mathcal{P}_k^p$ ,  $u^*TM$  is an  $L_{k;loc}^p$ -bundle, so we can talk about sections which are locally of  $L_{k;loc}^p$ -type. We shall introduce the shorthand  $L_k^p(u) = L_{k;loc}^p(u^*TM)$  and we may also consider

$$W_k^p(u) = \{\xi \in L_k^p(u) : \xi(s, 0) \in T_{u(s,0)}L_0, \xi(s, 1) \in T_{u(s,1)}L_1\},$$

so here we have tangent pointing along the Lagrangian boundary.

We can also consider spaces of sections  $W_l^q(u)$  and  $L_l^q(u)$  of different regularity provided that  $l \leq k$  and

$$(4.1) \quad l - \frac{2}{q} \leq k - \frac{2}{p}.$$

**Theorem 4.3.** ([8, Theorem 3]) *Let  $p \geq 1$  and  $kp > 2$ . Then  $\mathcal{P}_k^p(x, y)$  is a smooth Banach manifold and its tangent space at  $u$  is given by  $T_u \mathcal{P}_k^p(x, y) = W_k^p(u)$ .*

To show this is a Banach manifold, Floer uses a system of charts based on the exponential map. Accordingly, pick a family of metrics  $(g_t)$  such that  $L_k$  is totally geodesic with respect to  $g_k$ , as in Lemma 4.2.

Define

$$\begin{aligned} \exp : \Sigma \times TM &\rightarrow M, \\ \exp(s, t, x, v) &= \exp_{g_t}(x, v). \end{aligned}$$

Let  $\iota$  denote the minimal injectivity radius of the metrics  $g_t$  and define

$$U_u = \{\xi \in W_k^p(u) : \|\xi\|_\infty < \iota\}.$$

On a noncompact manifold  $M$ , we will not necessarily have  $\iota > 0$ . However, in our cases, this will hold since all our symplectic manifolds are geometrically bounded at infinity.

The charts are now given by

$$\begin{aligned} \exp_u : U_u &\rightarrow V_u = \exp_u(U_u), \\ \exp_u(\xi)(s, t) &= \exp(s, t, u(s, t), \xi(s, t)). \end{aligned}$$

It is because of this system of charts that we restricted the convergence conditions at the ends in defining  $\mathcal{P}_k^p(x, y)$ . The proof of above theorem is technical but makes no use of the symplectic structure.

Moreover we may also consider Banach bundles  $\mathcal{W}_l^q \rightarrow \mathcal{P}_k^p(x, y)$  and  $\mathcal{L}_l^q \rightarrow \mathcal{P}_k^p(x, y)$ , with fibres modelled on  $W_l^q(u)$  and  $L_l^q(u)$  respectively, provided that the regularity condition (4.1) holds.

The same proof as in [8] shows that  $\bar{\partial}_{\mathbf{J}}$  is a smooth section of  $\mathcal{L}_{k-1}^p$ . In fact, since  $\bar{\partial}_{\mathbf{J}}$  is a real Cauchy-Riemann operator with totally real boundary conditions [19, Appendix C]  $\bar{\partial}_{\mathbf{J}}$  is a Fredholm operator. We denote its linearization at  $u$  by  $E_u = D\bar{\partial}_{\mathbf{J}}(u): W_k^p \rightarrow L_{k-1}^p$ .

We now consider the zero-set of the section  $\bar{\partial}_{\mathbf{J}}$ . It is shown in [25] that if  $u \in \mathcal{M}(x, y)$ , then  $u$  has the right convergence conditions at the ends to be an element of  $\mathcal{P}_k^p(x, y)$  and moreover these sets are locally homeomorphic. Moreover, any solution to  $\bar{\partial}_{\mathbf{J}}u = 0$  will in fact be smooth, using elliptic bootstrapping techniques. This is proved in [8] for  $\omega$ -compatible  $\mathbf{J}$ , and this proof carries over in region  $U_x$ , and elsewhere it follows from [19, Proposition 3.1.9]. Therefore the zero set of  $\bar{\partial}_{\mathbf{J}}$  is precisely  $\cup_{x,y} \mathcal{M}_{\mathbf{J}}(x, y)$ .

**4.3. Fredholm theory.** This zero set will not always be a manifold, but we shall show that we can always perturb  $\mathbf{J} = (J_t)$  to some arbitrarily close  $\tilde{\mathbf{J}} = (\tilde{J}_t)$  such that the corresponding moduli space  $\mathcal{M}_{\tilde{\mathbf{J}}}$  is in fact a manifold. To do this, we need to have some space which represents the possible perturbations of  $\mathbf{J}$ .

The space of  $\omega$ -tame  $J$  is a Fréchet manifold whose tangent space at  $J$  is given by smooth sections of  $\text{End}(TM, J, \omega)$ , which is defined to be the bundle over  $M$  whose fibre at  $x$  is the space of linear maps  $Y: T_x M \rightarrow T_x M$  such that  $YJ + JY = 0$ . In order that we may have a Banach manifold, not a Fréchet one, we use the following argument of Floer [8].

Pick any sequence of positive real numbers  $(\epsilon_k)$  and define

$$\|Y\|_{\epsilon} = \sum \epsilon_k \max_x |D^k Y(x)|.$$

Denote by  $C_{\epsilon}^{\infty}(M, \text{End}(TM, J, \omega))$  those  $Y$  with finite  $\|\cdot\|_{\epsilon}$  norm. This is a Banach manifold. Floer [8] proves that there is a sequence  $(\epsilon_k)$  that tends to zero sufficiently quickly that  $C_{\epsilon}^{\infty}(M, \text{End}(TM, J, \omega))$  is dense in  $L^2(M, \text{End}(TM, J, \omega))$ .

Now fix some 1-parameter family  $\mathbf{J}^0 = (J_t^0)$  of almost complex structures. For a 1-parameter family  $\mathbf{Y} = (Y_t)$  of elements of  $C_{\epsilon}^{\infty}(M, \text{End}(TM, J, \omega))$ , we consider the map  $f: Y_t \mapsto J_t^0 \exp(-J_t^0 Y_t)$ . On some neighbourhood of the zero-section  $f$  restricts to a diffeomorphism. Define

$$\mathcal{Y} = \{\mathbf{Y} = (Y_t): \|Y_t\|_{\infty} < r \text{ and } Y_t(p) = 0 \text{ for } p \in U\},$$

where  $U = \cup_x U_x$  is our neighbourhood of the intersection points  $x$  and  $r$  is chosen small enough such that the restriction of  $f$  is a diffeomorphism. Denote by  $\mathcal{J}_r(\mathbf{J}^0)$  the image of  $\mathcal{Y}$  under  $f$ . This space represents our space of perturbations of  $\mathbf{J}^0$ . In what follows, we shall usually consider  $\mathbf{J}^0$  to be fixed and write  $\mathcal{J}$  instead of  $\mathcal{J}_r(\mathbf{J}^0)$ .

We have a section of Banach manifolds

$$\begin{aligned}\tilde{\partial}: \mathcal{P} \times \mathcal{Y} &\rightarrow \mathcal{L}, \\ \tilde{\partial}(u, \mathbf{Y}) &= \bar{\partial}_{f(\mathbf{Y})}u.\end{aligned}$$

As before, this section is smooth. We want to prove that its linearization is surjective on its zero set. Since  $E_u = D\tilde{\partial}_{\mathbf{J}}(u)$  is closed, it suffices to prove that the image is dense whenever  $\tilde{\partial}_{\mathbf{J}}u = 0$ . This is proved in the Appendix of [22], which is itself a correction of the argument appearing in [8]. This result makes no assumption of any  $\omega$ -compatibility condition.

Now the implicit function theorem [19, Theorem A.3.3] says that the *universal Floer moduli space*

$$\mathcal{M}(x, y, \mathcal{J}) = \{(u, \mathbf{J}) : u \in \mathcal{M}_{\mathbf{J}}(x, y)\}$$

is a smooth Banach manifold. Once we have this, we may consider the projection onto the  $\mathcal{J}$  factor, which is a Fredholm map and apply the Sard-Smale theorem.

**Theorem 4.4** (Sard-Smale). *The set of regular values of a Fredholm map  $g: A \rightarrow B$  between paracompact Banach manifolds is a Baire set in  $B$ .*

This shows that there is a second category set  $\mathcal{J}_{reg} \subset \mathcal{J}$  of so-called *regular* almost complex structures, such that  $\mathcal{M}_{\mathbf{J}}$  is a smooth manifold for  $\mathbf{J} \in \mathcal{J}_{reg}$ . In particular, this means that there exist regular  $\mathbf{J}$  arbitrarily close to  $\mathbf{J}^0$ . The dimension of this manifold is given by the Fredholm index, which in this case is  $|x| - |y|$ , the difference of the Maslov indices of the intersections [7]. Note also that  $\mathcal{M}_{\mathbf{J}}(x, y)$  carries a free  $\mathbb{R}$ -action by translation in the  $s$  variable and we shall denote the quotient space by  $\widehat{\mathcal{M}}_{\mathbf{J}}(x, y)$ .

**4.4. Compactifications.** From this point onward we shall assume that  $c_1(M) = 0$ . This is independent of the almost complex structure chosen. From the previous section, we now know that, given two intersection points  $x$  and  $y$ ,  $\mathcal{M}_{\mathbf{J}}(x, y)$  is a smooth manifold of the correct dimension, provided we pick  $\mathbf{J} \in \mathcal{J}_{reg}$ . Given some real number  $E$ , we can restrict attention to the set  $\mathcal{M}_{\mathbf{J}}^E(x, y)$  of Floer trajectories with the energy bound  $E(u) < E$ . Gromov compactness says that this manifold admits a natural compactification by adding broken trajectories, possibly with bubbles. In order to be able to define Floer cohomology, we shall need to look at the compactifications of these moduli spaces in cases when they have dimension  $\leq 2$ .

We want to prove that we can pick our almost complex structures  $(J_t)$  in such a way that we get no bubbling along solutions to the Floer equation. There are two possible types of bubbles: discs appearing on the Lagrangian boundary, and spheres appearing on the interior of some Floer solution. We shall prove that in the case where  $c_1(M) = 0$ , we can exclude the possibility of sphere bubbles. Disc bubbles are more difficult and no general approach exists to deal with these (in fact such an approach *cannot* exist in all situations as evidenced by the existence of *obstructed* Lagrangians [10]). However, we shall show later that we can avoid such bubbles in some specific cases. To prove that we get no sphere bubbles, we adapt the argument found in [13].

Fix some nonzero homology class  $A \in H_2(M; \mathbb{Z})$ . For a given  $J$ , we can consider the moduli space of simple  $J$ -holomorphic maps  $v: S^2 \rightarrow M$  representing the homology class  $A$ , which we shall denote  $\mathcal{M}_s(A, J)$ . We can also take a 1-parameter family  $\mathbf{J} = (J_t)$  and consider the space

$$\mathcal{M}_s(A, \mathbf{J}) = \{(t, v) : v \in \mathcal{M}_s(A, J_t)\}.$$

We can also consider the universal moduli space

$$\mathcal{M}_s(A, \mathcal{J}) = \{(t, v, \mathbf{J}) : (t, v) \in \mathcal{M}_s(A, \mathbf{J})\}.$$

This is a smooth Banach bundle and the projection to  $\mathcal{J}$  is Fredholm of index  $2n + 2c_1(A) + 1$ , so that for  $\mathbf{J} \in \mathcal{J}'_{reg}$  some second category set of almost complex structures,  $\mathcal{M}_s(A, \mathbf{J})$  is a smooth manifold of that dimension. The analysis underlying all this is similar to that in the previous section and can be found, for example, in [19]. We also note that  $\mathcal{M}_s(A, \mathbf{J})$  admits a free action by the real 6-dimensional reparametrization group of the sphere  $G = PSL(2, \mathbb{C})$  and we consider the space  $\mathcal{M}_s(A, \mathbf{J}) \times_G S^2$ , which, for generic  $\mathbf{J}$ , is a smooth manifold of dimension  $2n + 2c_1(M) - 3$ .

By taking the fibre product over  $\mathcal{J}$ , we can consider

$$\mathcal{N} = (\mathcal{M}_s(A, \mathcal{J}) \times_G S^2) \times_{\mathcal{J}} (\mathcal{M}(x, y, \mathcal{J}) \times [0, 1])$$

and the map

$$\mathcal{N} \rightarrow M \times [0, 1] \times M \times [0, 1]$$

given by

$$([v, z], t, u, t') \mapsto (v(z), t, u(0, t'), t').$$

We want to know the intersection of the image of this map with the diagonal  $\Delta_{M \times [0, 1]}$ . Since  $\mathcal{M}_{\mathbf{J}}(x, y)$  carries an  $\mathbb{R}$ -action, if there is any such intersection, there must be a bubble intersecting a Floer solution  $u$  at some  $u(0, t)$ , since we only care about  $J_{t_0}$ -bubbles meeting some Floer solution at time  $t_0$ .

For any  $t$ , we have an evaluation map  $ev_t: \mathcal{M}_{\mathbf{J}}(x, y) \rightarrow M$  given by  $ev_t(u) = u(0, t)$  and a version of Proposition 3.4.2 in [19] says that this map is a submersion for all  $t$ . This means that the intersection with the diagonal is transverse, and therefore the space

$$\mathcal{Z} = \{([v, z], t, u, t') : (v(z), t) = (u(0, t'), t')\}$$

is a submanifold of  $(\mathcal{M}_s(A, \mathcal{J}) \times_G S^2) \times_{\mathcal{J}} (\mathcal{M}(x, y, \mathcal{J}) \times [0, 1])$  of codimension  $2n + 1$ . This means that the projection  $\mathcal{Z} \rightarrow \mathcal{J}$  has Fredholm index

$$\begin{aligned} & (2n + 2c_1(A) - 3) + (|x| - |y| + 1) - (2n + 1) \\ &= 2c_1(A) + |x| - |y| - 3. \end{aligned}$$

Since we have  $c_1 = 0$ , this means that for generic  $\mathbf{J} = (J_t)$ , the 1- and 2-dimensional moduli spaces of Floer solutions (which are needed to define the Floer differential  $d$  and show that  $d^2 = 0$ ) will not intersect any sphere bubbles. Bearing in mind that the compactification of these spaces involves adding broken solutions, possibly with bubbles, the same argument as in [13] shows that we still get no intersection even after compactifying our spaces.

The case of disc bubbles is more difficult and there is no general approach that will work, but if we have chosen appropriate  $J_0, J_1$  such that we get no disc bubbles

for our Lagrangians, then picking a generic path of almost complex structures  $(J_t)$  interpolating between these two gives a family of  $(J_t)$  such that we can in fact define  $HF(L_0, L_1)$ . This will be discussed more in Section 4.7.

**4.5. Floer cohomology.** We first fix the coefficient field we shall use. Although (subject to certain topological assumptions) the relevant moduli spaces can be oriented so that Floer cohomology can be defined over fields of arbitrary characteristic, we don't need this for our purposes. We therefore introduce the Novikov ring

$$\Lambda_{\mathbb{Z}/2} = \left\{ \sum_r a_r q^r : a_r \in \mathbb{Z}/2, r \in \mathbb{R}, r \rightarrow \infty, \#\{a_r \neq 0 : r < E\} < \infty \text{ for all } E \right\}$$

of power series in the formal parameter  $q$  as in the Introduction. This is in fact a field.

In order to define Floer cohomology, we define the Floer cochain complex to be

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \Lambda_{\mathbb{Z}/2} \langle x \rangle.$$

In the case where  $|y| = |x| - 1$ , the Floer differential is defined by

$$dy = \sum_{u \in \widehat{\mathcal{M}}_{\mathbf{J}}(x, y)} q^{E(u)} x.$$

For this map to be well-defined over the Novikov ring, for any  $E$ , there must be only finitely many terms involving powers of  $q$  less than  $E$ . This follows from Gromov compactness. When  $|y| = |x| - 1$ , the compactification of  $\widehat{\mathcal{M}}_{\mathbf{J}}^E(x, y)$  can only involve adding bubbles, since breaking cannot occur as the solutions are already of minimal index. But we have shown that we can pick  $\mathbf{J}$  such that no bubbling occurs. Therefore the zero-dimensional manifold  $\widehat{\mathcal{M}}_{\mathbf{J}}^E(x, y)$  is compact, hence consists of finitely many points.

In order to show that this is in fact a differential (i.e. that  $d^2 = 0$ ), the standard approach is to identify the boundary of the compactification of any 1-dimensional  $\widehat{\mathcal{M}}_{\mathbf{J}}(x, z)$  with  $\widehat{\mathcal{M}}_{\mathbf{J}}(x, y) \times \widehat{\mathcal{M}}_{\mathbf{J}}(y, z)$ , and use the fact that boundary points of a 1-manifold come in pairs. This identification again relies on the fact that no bubbling occurs, which is ensured by the previous section. Once again we stress that we have not yet dealt with disc bubbling, so that the content of this section is incomplete and Floer cohomology will not be properly defined until we do so in Section 4.7.

In our setting, where  $c_1(M) = 0$ , we may also pick a grading so that  $HF^*(L_0, L_1)$  becomes a  $\mathbb{Z}$ -graded group [26].

We also want to define a multiplication map on Floer cohomology. We start by doing this on the chain level.

Consider three Lagrangian submanifolds  $L_i$ ,  $i = 0, 1, 2$  and transverse intersection points  $x \in L_0 \cap L_2$ ,  $y \in L_0 \cap L_1$ ,  $z \in L_1 \cap L_2$ . Similar to before we may consider the moduli space  $\mathcal{M}_{\mathbf{J}}^2(x, y, z)$  of holomorphic curves  $u$  from a disc with 3 marked boundary points mapping to  $M$  such that the marked boundary points tend to  $x, y, z$  and the remainder of the boundary maps to the various Lagrangians (see [28, Section 2] for more specific details). Here  $\mathbf{J}$  is a 2-parameter family of almost

complex structures  $(J_w)_{w \in \mathbb{D}}$  and a similar analysis to the previous section shows that, for a generic choice of  $\mathbf{J}$ ,  $\mathcal{M}_{\mathbf{J}}^2(x, y, z)$  is a smooth manifold of dimension  $|x| - |y| - |z|$ .

We can therefore define

$$m: CF(L_1, L_2) \otimes CF(L_0, L_1) \rightarrow CF(L_0, L_2),$$

$$m(z, y) = \sum_{u \in \mathcal{M}_{\mathbf{J}}^2(x, y, z)} q^{E(u)} x.$$

in the case where  $|x| = |y| + |z|$ . We want this to be a chain map so that we get a multiplication on the cohomological level.

Here the standard approach is again to consider the boundary of the compactification of the 1-dimensional part of  $\mathcal{M}_{\mathbf{J}}^2(x, y, z)$  (see for example [23]). However, in our case we must once more rule out the possibility of bubbling off of spheres (disc bubbles will be dealt with in Section 4.7).

We continue in a similar vein to before and consider the universal moduli space

$$\mathcal{M}^2(x, y, z, \mathcal{J}) = \{(u, \mathbf{J}) : u \in \mathcal{M}_{\mathbf{J}}^2(x, y, z)\}$$

for an appropriate Banach space  $\mathcal{J}$  of 2-parameter families of almost complex structures defined similarly to the previous section. We then consider

$$\mathcal{N}' = (\mathcal{M}_s(A, \mathcal{J}) \times_G S^2) \times_{\mathcal{J}} (\mathcal{M}^2(x, y, z, \mathcal{J}) \times \mathbb{D}).$$

By mapping to  $M \times \mathbb{D} \times M \times \mathbb{D}$  via  $([v, z], w, u, w') \mapsto (v(z), w, u(w'), w')$ , we see that  $\mathcal{N}'$  contains a submanifold

$$\mathcal{Z}' = \{([v, z], w, u, w') : (v(z), w) = (u(w'), w')\}$$

of codimension  $2n + 2$ , which represents the intersections between  $J_w$ -bubbles and multiplication curves  $u$  at point  $u(w)$ . The projection  $\mathcal{Z}' \rightarrow \mathcal{J}$  is Fredholm of index

$$(4.2) \quad (2n + 2c_1(A) - 2) + (|x| - |y| - |z| + 2) - (2n + 2) \\ = 2c_1(A) + |x| - |y| - |z| - 2.$$

Therefore, for generic  $\mathbf{J} = (J_w)$ , the 0- and 1-dimensional moduli spaces of such holomorphic discs do not intersect any sphere bubbles (recall that we are assuming  $c_1(M) = 0$ ), so these will not obstruct our multiplication surviving to cohomology.

We shall also want, when defining *wrapped* Floer cohomology, to have a map

$$\Psi_H: CF(L_0, L_1) \rightarrow CF(L_0, \psi_H(L_1)),$$

where  $\psi_H$  is the Hamiltonian isotopy coming from some Hamiltonian  $H: M \times [0, 1] \rightarrow \mathbb{R}$  (when  $M$  is noncompact but convex at infinity, we additionally require  $H$  to be *monotone*:  $\partial_s H_s \leq 0$  [23]). First note that intersection points  $y \in L_0 \cap \psi(L_1)$  are in one-to-one correspondence with *Hamiltonian chords*  $y: [0, 1] \rightarrow M$  such that  $y(0) \in L_0, y(1) \in L_1$ , and  $\dot{y}(s) = X_H(y(s))$ .

For  $x \in L_0 \cap L_1$  and  $y \in L_0 \cap \psi(L_1)$ , we consider the moduli space of *continuation Floer trajectories*  $\mathcal{M}_{\mathbf{J}}^H(x, y)$ , solutions  $u$  to the equation

$$\partial_s v + J_{s,t}(\partial_t v - X_H) = 0$$

on the strip  $\mathbb{R} \times [0, 1]$  such that  $u(\cdot, 0) \in L_0$ ,  $u(\cdot, 1) \in L_1$ , and which converge to the point  $x$  at  $+\infty$  and to the chord  $y(t)$  at  $-\infty$ . The standard approach [2] shows that, for generic  $\mathbf{J} = (J_{s,t})$ , this moduli space is a smooth manifold of dimension  $|y| - |x|$  and we can define

$$\Psi_H x = \sum_{u \in \mathcal{M}_{\mathbf{J}}^H(x, y)} q^{E(u)} y$$

in the case when  $|y| = |x|$ . Again the standard argument involving the 1-dimensional part of  $\mathcal{M}_{\mathbf{J}}^H(x, y)$  shows that this is a chain map modulo bubbling. But no bubbling of spheres occurs because of the same dimension count as in (4.2) replacing  $\text{vdim } \mathcal{M}_{\mathbf{J}}^2(x, y, z)$  with  $\text{vdim } \mathcal{M}_{\mathbf{J}}^H(x, y)$ : the space  $\mathcal{Z}''$  representing intersections between  $J_{s,t}$ -bubbles and continuation trajectories at  $u(s, t)$  has virtual dimension

$$\begin{aligned} & (2n + 2c_1(A) - 2) + (|y| - |x| + 2) - (2n + 2) \\ &= 2c_1(A) + |y| - |x| - 2. \end{aligned}$$

Note that we are here using 2-parameter families of almost complex structures on  $\mathbb{R} \times [0, 1]$  as opposed to the 1-parameter families used in defining  $d$ . See Section 4.7 for the argument for disc bubbles.

A similar argument shows that  $\Psi_H$  intertwines the multiplicative structures on  $HF(L_0, L_1)$  and  $HF(L_0, \psi_H(L_1))$ .

**Remark 4.5.** *In the case of exact Lagrangians in an exact symplectic manifold, much of the above analysis is unnecessary: exactness means that no bubbles occur in the compactifications of our moduli spaces, and we also get a priori energy bounds independent of  $u$ , so we can actually work over  $\mathbb{Z}/2$  should we wish.*

**4.6. Floer cohomology in Lefschetz fibrations.** In the context of a Lefschetz fibration  $\pi: E \rightarrow \mathbb{C}$ , we can make a choice of almost complex structures which lends itself well to Floer cohomology calculations.

In some neighbourhood of  $E^{crit}$  we pick  $J$  to agree with the standard integrable complex structure in the local model  $z \mapsto \sum z_i^2$  as in Definition 2.1, which makes  $\omega$  locally a Kähler form. Away from  $E^{crit}$ , we have the splitting

$$T_x E = T_x^h E \oplus T_x^v E$$

where  $T_x^v E = \ker(D\pi_x)$  and  $T_x^h E \cong T_{\pi(x)} \mathbb{C}$ . With respect to this splitting, we choose  $J$  that, away from  $E^{crit}$ , look like

$$\begin{pmatrix} j & 0 \\ 0 & J^v \end{pmatrix},$$

such that  $J^v$ , the vertical part of  $J$ , is compatible with  $\omega$  restricted to the fibre and  $j$  is compatible with the standard form on the base. Such a  $J$  makes the projection  $\pi$   $J$ -holomorphic, so that Floer solutions in  $E$  project to  $j$ -holomorphic strips  $\pi \circ u: \Sigma \rightarrow \mathbb{C}$ , and we can now use the maximum principle for holomorphic functions to restrict the region in which Floer solutions may appear.

The problem is that such a  $J$  will not necessarily be regular, so not be suitable for defining  $HF(L_0, L_1)$ . In [17], they proceed as follows. They take some small generic

perturbation of  $(J_t)$  to regular  $(\tilde{J}_t)$  such that  $(\tilde{J}_t)$  is still  $\omega$ -compatible, losing in the process the property that  $\pi$  is holomorphic. However, Gromov compactness says that Floer solutions for  $(J_t)$  will be close to Floer solutions for  $(\tilde{J}_t)$ . In order to apply Gromov's compactness theorem for this argument to work, we need some energy bounds, which a priori exist in the setting of [17] as all their manifolds are exact.

We do not have any such energy bounds. Therefore, we perturb  $J$  by adding some *horizontal* component to get

$$\tilde{J} = \begin{pmatrix} j & 0 \\ H & J^v \end{pmatrix},$$

where  $H$  is some small perturbation that is zero on some neighbourhood of the intersections of our Lagrangians and such that  $\tilde{J}^2 = -1$ . Now  $\pi$  is still holomorphic, so we can use maximum principles in the base, but  $\tilde{J}$  is no longer compatible with  $\omega$ . However, for small  $H$ , it will still tame  $\omega$  and we can use the discussion above to say that we can still do Floer cohomology in this setting. The proof that the space of such  $H$  is large enough for us to achieve transversality as in Section 4.3 can be found in [27, Lemma 2.4].

**4.7. Disc bubbles.** We have not yet said anything about how to avoid disc bubbles,  $J$ -holomorphic maps  $w: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$ . However, for the purposes of this paper, we need only consider specific sorts of Lagrangian submanifolds, namely spheres or Lefschetz thimbles in some Lefschetz fibration, with a six-dimensional total space and whose first Chern class vanishes.

In the first instance, it is shown in [32, Corollary 4.5], using techniques inspired by symplectic field theory, that for a Lagrangian sphere  $L$  in a symplectic manifold of dimension at least 4 with vanishing first Chern class, there exists a  $J_L$  such that the Floer cohomology of  $L$  is unobstructed ( $(L, J_L)$  is an *elementary Lagrangian conductor* in the language of Welschinger) and moreover we have the classical isomorphism  $HF^*(L, L) \cong H^*(S^n, \Lambda_{\mathbb{Z}/2})$  [32, Corollary 4.12]. This is proven in [32] only for compatible  $J$ , not the larger class of almost complex structures we have considered in this section. However, in the next section, there is only one point at which we need to consider the Floer cohomology of a 3-sphere in the total space of a Lefschetz fibration (Section 5.1) and here we don't need to perform the horizontal perturbation trick, so at this point in the argument we can just pick a compatible  $J$  for the sphere as usual.

As for thimbles, we start by picking  $J$  adapted to our Lefschetz fibration as above. If a disc bubble exists, then by considering the projection to the base, we see that any such bubble must entirely be contained in some fibre of  $\pi: E \rightarrow \mathbb{C}$ . The part of the thimble living in this fibre is now just a sphere, so we can arrange for the vertical part  $J^v$  of  $J$  to be such that we get no bubbles as in the previous paragraph. However, this fails to take into account of the fact that we have a 1-parameter family of such fibres (the vanishing path). In fact, in [32] the relevant Fredholm problem involves a Fredholm operator whose index is bounded from above by  $-2$ , so we may in fact generically pick a 1-parameter family of such  $J$  so that the Floer cohomology is unobstructed.

Now to complete the definition of the Floer cohomology of two such Lagrangians, we pick appropriate  $J_0$  and  $J_1$  as above and then pick some path  $\mathbf{J} = (J_t)$  interpolating between them. A generic perturbation of  $\mathbf{J}$ , which may be chosen such that the endpoints are fixed will then be suitable. We may do likewise to exclude the possibility of disc bubbles appearing in the compactifications of  $\mathcal{M}_{\mathbf{J}}^2(x, y, z)$  and  $\mathcal{M}_{\mathbf{J}}^H(x, y)$  (although we now consider 2-parameter families of almost complex structures, we are free to choose that  $\mathbf{J}$  be constant along the boundary components of the disc/strip since we can achieve transversality by perturbing  $\mathbf{J}$  just on the interior), thus completing the constructions of Section 4.5.

**Remark 4.6.** *Welschinger [32] establishes a result saying that, given a Lagrangian sphere  $L$  and any  $E > 0$ , there exists a second category set of almost complex structures  $J_E$  such that  $E(w) > E$  for any  $J_E$ -holomorphic disc  $w: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$ . These bubbles can then be discounted by using the Novikov ring. A similar sort of argument is perhaps best explained in [10, Chapter 4.6].*

Briefly, suppose that we pick almost complex structures  $J_i$  such that any  $J_i$ -holomorphic disc  $w$  has energy  $E(w) > i$ . We can construct, for each  $i$ ,  $A_\infty$ -structures  $\{\mu_d^i\}$  on the space of cochains  $C^*(L)$  where, by assumption,  $\{\mu_d^i\} = 1 + O(q^i)$ . (Here  $q$  is our formal Novikov parameter.) In [10, Chapter 4.6], they construct  $A_\infty$ -functors  $\mathcal{F}^i: (C^*(L), \{\mu_d^i\}) \rightarrow (C^*(L), \{\mu_d^{i+1}\})$  which come from counts of genus 0 stable curves, all of whose components are  $J_\alpha$ -holomorphic for some  $i \leq \alpha \leq i+1$ . Again,  $\mathcal{F}^i = \{\mathcal{F}_r^i\} = \text{Id} + O(q^i)$  in our Novikov filtration. This means that  $\prod_{i=1}^\infty \mathcal{F}^i$  converges over  $\Lambda_{\mathbb{Z}/2}$ , and so defines an  $A_\infty$ -functor from  $(C^*(L), \{\mu_d^1\})$  to the classical  $A_\infty$ -structure on  $C^*(L)$ . We may then pull back the classical Maurer-Cartan solution for which  $HF^*(L, L) \cong H^*(S^n, \Lambda_{\mathbb{Z}/2})$  by  $\prod_{i=1}^\infty \mathcal{F}^i$ .

## 5. THE EXAMPLES OF MAYDANSKIY-SEIDEL

Using the same method as explained in Section 2, we can construct the six-dimensional symplectic manifold  $X_2$  in Figure 5.1. Its generic fibre is diffeomorphic to the  $A_{m+1}$  Milnor fibre  $M_{m+1}$  and the Lefschetz fibration  $\pi: X_2 \rightarrow \mathbb{C}$  has  $m+1$  critical points corresponding to  $m+1$  vanishing cycles in  $M_{m+1}$ . The first  $m$ ,  $V_1, \dots, V_m$  come from the straightline matching paths, but  $V_{m+1}$  is the sphere associated to the curved path  $\gamma_{m+1}$ . For each critical value  $x_i$ , corresponding to  $V_i$ , fix some vanishing path  $\beta_i: [0, \infty) \rightarrow \mathbb{C}$  such that  $\beta_i(t) = t$  for  $t \gg 0$ . Let  $\Delta_i \subset X_2$  denote the corresponding Lefschetz thimble.

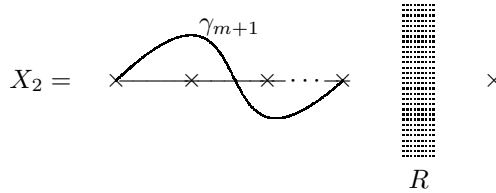


FIGURE 5.1.

A trivial extension of the argument in [17], which will be recapped in this section, shows that  $X_2$  is diffeomorphic to  $T^*S^3 \cup 2\text{-handle}$  and also contains no Lagrangian

sphere  $L$  such that  $[L] \neq 0$  in  $H_2(X_2; \mathbb{Z}/2)$ . (We have shown below only one such possible choice of  $\gamma_{m+1}$ ; there are infinitely many others for which this is also true [17].) We construct a deformation  $\tilde{X}_2$  of this manifold by adding on a closed 2-form supported in the shaded region  $R$ , as in Section 3, to obtain a family of symplectic manifolds  $(\tilde{X}_2^t, \omega_t)$ .  $c_1(X_2) = 0$  so therefore  $c_1(\tilde{X}_2^t) = 0$  for all  $t$ . We also note that, after deformation, the  $V_i$  will still be Lagrangian in  $M_{m+1}$  since they live away from the region  $R$ . Also the thimbles  $\Delta_i$  will stay Lagrangian in  $\tilde{X}_2^t$ .

In this section, we shall prove the following:

**Theorem 5.1.** *For all  $t \in [0, 1]$ ,  $\tilde{X}_2^t$  contains no Lagrangian sphere  $L$  such that  $[L] \neq 0 \in H_2(\tilde{X}_2^t; \mathbb{Z}/2)$ .*

The proof of this will essentially just be a repeat of the argument in [17], so we shall not explain all the details fully, instead directing the interested reader to the relevant sections of [17]. However, this proof relies heavily on the technology of Floer cohomology and Fukaya categories. In the original paper, everything is carried out working within the category of exact symplectic manifolds so the analytical issues involved in setting up Floer cohomology are easily overcome. This was why we had to go through the analysis of the previous section as we now often have to work in the more problematic nonexact setting. With the results of the previous section however, the argument of [17] more or less just carries over, and we only make a few remarks where particular care needs to be exercised.

In what follows, we shall denote by  $HF_t^*(L_0, L_1)$  the Floer cohomology computed with respect to  $\omega_t$  in any situations where there is likely to be confusion about the symplectic form being used.

**5.1. Wrapped Floer cohomology.** We start by defining a variant of Floer cohomology, wrapped Floer cohomology. Following [17], we shall not need to define this in the level of generality found in [2, 23], but instead restrict to a simpler (and, in our setting, equivalent) definition which is well-suited to Lefschetz fibrations.

Given a Lefschetz fibration  $\pi: E \rightarrow \mathbb{C}$ , we consider a Hamiltonian  $H: E \rightarrow \mathbb{R}$  of the form  $H(y) = \psi(\frac{1}{2}|\pi(y)|^2)$  where  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  is such that  $\psi(r) = 0$  for  $r < 1/2$  and  $\psi'(r) = 1$  for  $r \gg 0$ . Let  $\Phi^\alpha$  denote the time- $\alpha$  flow of this Hamiltonian and, given some Lagrangian  $L$ , we define  $L^\alpha = \Phi^\alpha(L)$ .

We can now define the wrapped Floer cohomology of a Lagrangian  $L$  and a thimble  $\Delta$  (where, in order to exclude bubbling of discs as mentioned previously,  $L$  is either a sphere or another thimble) to be the direct limit of Floer cohomology groups

$$HW_t^*(L, \Delta) = \varinjlim_k HF_t^*(L, \Delta^{2\pi k + \epsilon})$$

for some very small  $\epsilon > 0$ . The maps involved in this direct limit are the continuation maps from Section 4.5.

We will actually need to perform an extra small Hamiltonian isotopy in addition to  $\Phi^\alpha$  in order to ensure transversality of intersections but will suppress further mention of this. For our purposes, it is not necessary to identify our Floer groups canonically so the details of how we do this are irrelevant for what follows.

To prove Theorem 5.1, suppose for sake of contradiction that there does exist a Lagrangian sphere  $L \subset \tilde{X}_2^t$  such that  $[L] \neq 0$  in  $H_*(\tilde{X}_2^t; \mathbb{Z}/2)$ .  $\tilde{X}_2^t$  is topologically  $T^*S^3$  with a 2-handle attached, and it is shown in [17, Section 9] that  $L \cdot \Delta_{m+1} \neq 0$  for such a sphere. This intersection number is the Euler characteristic of the Floer cohomology group  $HF_t^*(L, \Delta_{m+1})$ . Given the compactness of  $L$ , this group is equal to the wrapped Floer cohomology group  $HW_t^*(L, \Delta_{m+1})$  (we may choose to start “wrapping” outside some compact set containing  $L$ ) and  $HW_t^*(L, \Delta_{m+1})$  is itself a module over the unital ring  $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$ , where the multiplication maps here are the images under the direct limit of the multiplication defined in Section 4.5. Thus we conclude

**Lemma 5.2.** *If such a Lagrangian sphere exists, then  $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$ .*

The rest of this section is devoted to proving that  $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) = 0$  to provide the required contradiction.

**5.2. From total space to fibre.** If we consider the directed system of groups used to define  $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$ , we see that each step introduces new intersection points as the path over which our wrapped Lefschetz thimble lives wraps round the base once more. Choose our family of almost complex structures  $(J_t)$  as in Section 4.6. In [17], they establish the existence of a spectral sequence computing the wrapped Floer cohomology of any two thimbles, which carries over in our setting in light of the discussion of Section 4. When we consider  $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$ , this spectral sequence yields the following long exact sequence

$$\begin{array}{ccc} HF_t^*(\Delta_{m+1}, \Delta_{m+1}^\epsilon) & \longrightarrow & HF_t^*(\Delta_{m+1}, \Delta_{m+1}^{2\pi+\epsilon}) \\ & \searrow \sigma & \downarrow \\ & & HF_t^*(\mu(V_{m+1}), V_{m+1}), \end{array}$$

where the bottom group is calculated in the fibre  $E_z$  and  $\mu$  denotes the outer monodromy of the Lefschetz fibration. Lemma 2.2 allows us to identify some particular fibre  $E_{z'}$  with the manifold  $M$  included in the data of this lemma. We may arrange that  $z = z'$ .

In particular, since the unit in  $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$  arises as the image of  $1 \in HF_t^*(\Delta_{m+1}, \Delta_{m+1}^\epsilon) = \Lambda_{\mathbb{Z}/2}$ , the map  $\sigma$  must be zero. By analysing the curves involved in defining the map  $\sigma$  [17, Section 5] and comparing to the maps involved in Seidel’s long exact sequence [27], we can, by Poincaré duality, identify the map  $\sigma$  with an element of  $HF_t^0(V_{m+1}, \mu(V_{m+1}))$ , which we shall also denote by  $\sigma$ .

**Lemma 5.3.** ([17, Proposition 5.1]) *If  $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$ , then  $\sigma$  vanishes.*

**5.3. Fukaya categories.** We now shift attention to the Fukaya category of the fibre  $\mathcal{F}(E_z)$ , and introduce two related categories.

The first is a directed  $A_\infty$ -category  $\mathcal{A}$ , which has as objects the finite set  $\{V_1, \dots, V_m\}$  and morphisms

$$\text{hom}_{\mathcal{A}}(V_i, V_j) = \begin{cases} (\mathbb{Z}/2)e_i & \text{for } i = j \\ (\mathbb{Z}/2)f_i & \text{for } i = j - 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the degrees are chosen to be  $|e_i| = 0$  and  $|f_i| = 1$ . This category is chosen to reflect the fact that we have an  $A_m$  configuration of Lagrangian spheres  $V_i \subset M_m$  coming from the straightline paths in Figure 5.1, where the only points of intersection are between adjacent spheres and the gradings can be chosen in a nice way. This determines the higher-order  $A_\infty$ -structure, namely that the only nontrivial higher products are given by  $\mu^2(e_i, e_i) = e_i$  and  $\mu^2(f_i, e_i) = f_i = \mu^2(e_{i+1}, f_i)$ .

The second variant of the Fukaya category we shall consider is the  $A_\infty$ -category  $\mathcal{B}$ , which is the subcategory of the Fukaya category  $\mathcal{F}(E_z)$  generated by the following collection of Lagrangian submanifolds

$$V_1, \dots, V_m, V_{m+1}, \tau_{V_m}(V_{m+1}), \tau_{V_{m-1}}\tau_{V_m}(V_{m+1}), \dots, \tau_{V_1} \dots \tau_{V_m}(V_{m+1}).$$

In [17], there is no need to restrict attention specifically to  $\mathcal{B}$  and we can happily work with the whole Fukaya category  $\mathcal{F}(E_z)$ , even though as above we do not strictly need to. However, all the objects in  $\mathcal{B}$  are disjoint from the region  $R$  where  $\omega_t$  is nonexact and we can use maximum principles to ensure that all pseudoholomorphic curves between these objects also do not enter the region  $R$ . This means there is no extra analysis to do in defining the  $A_\infty$ -category  $\mathcal{B}$  as we are essentially just in an exact setting.

In what follows, we shall also want to use Seidel's long exact sequence in Floer cohomology [27]. Part of the proof of this long exact sequence in [27] relies on a spectral sequence argument coming from a filtration on Floer cochain groups given by the symplectic action functional. Seidel needs to upgrade this  $\mathbb{R}$ -filtration to some  $\mathbb{Z}$ -subfiltration in order to show that a certain mapping cone is acyclic, which can be done since the action spectrum will be discrete for finitely many exact Lagrangians in an exact symplectic manifold. In  $\mathcal{B}$ , this argument remains valid since maximum principles mean that we are considering the same holomorphic curves with the same actions as in the exact case, although this approach would not work in general.

We can consider the “derived” versions of  $\mathcal{A}$  and  $\mathcal{B}$  defined via twisted complexes as  $D\mathcal{A} = H^0(Tw\mathcal{A})$  and  $D\mathcal{B} = H^0(Tw\mathcal{B})$  [28]. There is a canonical (up to quasi-isomorphism) functor  $\iota : \mathcal{A} \rightarrow \mathcal{B}$  which on the derived level extends to an exact functor  $D\iota : D\mathcal{A} \rightarrow D\mathcal{B}$ .

On the level of derived Fukaya categories  $D\mathcal{B}$ , thanks to the result of Seidel [28] relating algebraic and geometric twisting operations,  $\sigma$  corresponds to an element  $S \in \text{hom}_{D\mathcal{B}}(V_{m+1}, T_{V_1} \dots T_{V_m} V_{m+1})$ . If  $\sigma$  vanishes  $S$  must too, so, looking at exact triangles in  $D\mathcal{B}$ , this means that

$$V_{m+1}[1] \oplus T_{V_1} \dots T_{V_m} V_{m+1} \cong \text{Cone}(S),$$

so we wish to understand  $C = \text{Cone}(S)$ .

Given all this, the next lemma is pure algebra.

**Lemma 5.4.** ([17, Proposition 6.2]) *If  $S = 0$ , then  $V_{m+1}$  is isomorphic to a direct summand of an object lying in the image of the functor  $D\iota: DA \rightarrow DB$ .*

**5.4. Contradiction.** The fibre  $E_z$  itself admits a Lefschetz fibration as pictured at the start of this section, such that the matching cycles of interest arise from matching paths  $\gamma_1, \dots, \gamma_{m+1}$ . By assumption,  $\gamma_{m+1}$  is not isotopic to  $\gamma_i$  for  $1 \leq i \leq m$  within the class of paths which avoid the critical values except possibly at their endpoints.

**Lemma 5.5.** ([17, Lemma 7.2]) *For  $1 \leq i \leq m$ , and for all  $t \in [0, 1]$ , the image of the product map*

$$HF_t^*(V_{m+1}, V_i) \otimes HF_t^*(V_i, V_{m+1}) \rightarrow HF_t^*(V_{m+1}, V_{m+1}) \cong H^*(V_{m+1}; \Lambda_{\mathbb{Z}/2})$$

*does not contain the identity in its image.*

As in [17], this is proved by considering the auxiliary Lagrangian  $L_\xi \cong S^1 \times \mathbb{R}$  associated to the path  $\xi$  in Figure 5.2. The key point is that, since  $\gamma_i$  is not isotopic to  $\gamma_{m+1}$ , we can draw  $\xi$  so that it intersects  $\gamma_{m+1}$  but is disjoint from  $\gamma_i$  (here we have drawn only two of the matching paths,  $\gamma_{m+1}$  and  $\gamma_i$ , to avoid clutter).

It is proven in [15] that  $\dim HF_t^*(L_\xi, V_{m+1}) > 0$ , whereas clearly we have  $\dim HF_t^*(L_\xi, V_i) = 0$ . As before, we may choose  $\xi$  to lie away from the region  $R$  where our deforming 2-form is supported since, by assumption, this also true for the paths  $\gamma_j$ , so once more we may use maximum principles to restrict all Floer solutions to a region of  $M_{m+1}$  where  $\omega_t$  is exact.

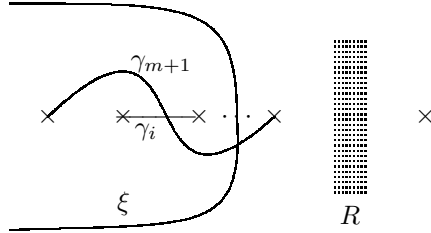


FIGURE 5.2.

Suppose we have elements  $a_1 \in HF_t^*(V_{m+1}, V_i)$  and  $a_2 \in HF_t^*(V_i, V_{m+1})$  such that  $a_2 \cdot a_1 \in H^0(V_{m+1})$ , the invertible part of this ring.

This then means that the composition

$$HF_t^*(L_\xi, V_{m+1}) \xrightarrow{a_1} HF_t^*(L_\xi, V_i) \xrightarrow{a_2} HF_t^*(L_\xi, V_{m+1})$$

is an isomorphism, which is a contradiction.

Once we have this, we can complete the proof of Theorem 5.1, the remainder of which carries over directly from [17] as it is essentially just algebra.

Suppose that  $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$ . Then  $V_{m+1}$  is contained in the image of  $D\iota: DA \rightarrow DB$ . Say that  $V_{m+1}$  occurs as a direct summand of  $C$  in the image.

Then, in particular

$$\mathrm{hom}_{D\mathcal{B}}(C, V_{m+1}) \otimes \mathrm{hom}_{D\mathcal{B}}(V_{m+1}, C) \rightarrow \mathrm{hom}_{D\mathcal{B}}(V_{m+1}, V_{m+1}) \cong H^*(S^n; \Lambda_{\mathbb{Z}/2})$$

contains the identity in its image as we can consider the maps corresponding to projection and inclusion with respect to this summand. However, thanks to the particularly simple form of  $\mathcal{A}$ , there exists a classification of twisted complexes in  $\mathcal{A}$ , following from Gabriel's theorem [11]. It says that any twisted complex is isomorphic to a direct sum of (possibly shifted copies of) the basic complexes  $C_{kl}$

$$C_{kl} = \begin{cases} W_i = \mathbb{Z}/2 & \text{for } k \leq i < l \text{ concentrated in degree 0} \\ W_i = 0 & \text{otherwise} \\ \delta_{i+1,i} = f_i & \text{for } k \leq i < l \\ \delta_{ij} = 0 & \text{otherwise.} \end{cases}$$

However, by repeated application of our Lemma 5.5 above, we derive a contradiction, since the terms in the  $C_{kl}$  correspond geometrically to  $V_i$  involved there. This completes the proof that  $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) = 0$ , and therefore, by Lemma 5.2, there cannot exist a homologically essential Lagrangian sphere in  $\tilde{X}_2^t$ .

## 6. DISTINGUISHING $X_1$ AND $X_2$

**6.1. Moser for symplectic manifolds convex at infinity.** Take a symplectic manifold  $(M, \omega)$  which is convex at infinity. Recall that this means that there is a relatively compact set  $M^{in}$  such that on a neighbourhood of the boundary  $\partial M^{in}$  we have a 1-form  $\theta$  such that  $d\theta = \omega$  and  $\theta|_{\partial M^{in}}$  is a contact 1-form, and that  $M \setminus M^{in}$  looks like the positive symplectization of  $\partial M^{in}$  according to  $\theta|_{\partial M^{in}}$ .

Suppose that we have a family of cohomologous 2-forms  $(\omega_t)_{t \in [0,1]}$  which make  $M^{in}$  a symplectic manifold with convex boundary. We can complete  $(M^{in}, \omega_t)$  to a family  $(M, \hat{\omega}_t)$  of noncompact symplectic manifolds with cohomologous symplectic forms all convex at infinity. We want to prove a version of Moser's theorem [21] in this setting.

**Lemma 6.1.** *The family  $(M, \omega_t)$  above are all symplectomorphic, by symplectomorphisms modelled on contactomorphisms at infinity.*

*Proof.* We follow the standard argument, but need to pay attention to possible problems arising from the noncompactness of  $M$ . Since the  $\omega_t$  are all cohomologous, we pick  $\sigma_t$  such that

$$\frac{d}{dt}\omega_t = d\sigma_t.$$

Then, Moser's theorem follows from considering the flow  $\psi_t$  defined by integrating the vector fields  $Y_t$  determined by

$$\sigma_t + \iota(Y_t)\omega_t = 0,$$

although we need to be careful that we can actually integrate  $Y_t$  all the way to time 1. This can be done because our forms are all cylindrical at infinity, so the vector fields obtained above will all scale according to  $e^r$  as we move in the  $r$ -direction along the collar. This bound is enough to ensure we can integrate to a flow.  $\square$

**6.2. Proof of Theorem 1.1.** To prove Theorem 1.1 we just apply Lemma 6.1 in our case. Let  $\omega_1, \omega_2$  be the exact forms induced on  $X_1, X_2$  respectively and suppose, for a contradiction, that there exists a diffeomorphism  $\phi: X_2 \rightarrow X_1$  such that  $\phi^*(\omega_1) = \omega_2$ .

Then we also consider the deforming 2-forms  $\eta_2$  and  $\phi^*(\eta_1)$  defined on  $X_2$  and by rescaling we may assume without loss of generality that these two 2-forms are cohomologous (since  $H^2(X_i; \mathbb{R}) = \mathbb{R}$ ). We now consider the family of 2-forms on  $X_2$

$$\Omega_t = (1-t)(\omega_2 + \eta_2) + t\phi^*(\omega_1 + \eta_1) = \omega_2 + t\phi^*(\eta_1) + (1-t)\eta_2.$$

There exists some compact subset  $X_2^{in}$  which is an interior for  $X_2$  with respect to  $\Omega_0 = \omega_2 + \eta_2$ , and by the compactness of both  $X_2^{in}$  and its boundary, we can say that, after perhaps once more rescaling  $\eta_1$  and  $\eta_2$  if necessary,  $\Omega_t$  makes  $X_2^{in}$  a symplectic manifold with convex boundary for all  $t$ . However,  $\Omega_t$  is not necessarily cylindrical for all  $t$  so we now change our family  $\Omega_t$ , by replacing  $\Omega_t|_{X_2^{out}}$  with the completion of  $\Omega_t|_{X_2^{in}}$  to get a new family of cohomologous symplectic forms  $\tilde{\Omega}_t$  on  $X_2^{in} \cup_{\partial X_2^{in}} [0, \infty) \times \partial X_2$ , which are all cylindrical on the collar. Therefore, by Lemma 6.1,  $(X_2^{in}, \tilde{\Omega}_t)$  are all symplectomorphic.

However, we can choose  $X_2^{in}$  sufficiently large that it contains the image  $\phi^{-1}(L)$  of the Lagrangian sphere exhibited in Section 3. This is a contradiction of Theorem 5.1.

## 7. SYMPLECTIC COHOMOLOGY VANISHES

In this section, we digress from the main theme and discuss symplectic cohomology. All symplectic manifolds considered in this section will be exact and we shall work with  $\mathbb{Z}/2$ -coefficients. As mentioned in the Introduction, symplectic cohomology is one of the standard invariants used to examine and distinguish Liouville domains. We prove that the symplectic cohomology  $SH^*(X_i; \mathbb{Z}/2)$  of  $X_1$  and  $X_2$  both vanish, thereby showing that this invariant does not suffice to distinguish between the examples of this paper, and so a different approach such as that of this paper truly is needed.

We shall not define symplectic cohomology here; an appropriate definition may be found in [30], for example. We shall instead refer to two results from [1]. In the formulation of these two lemmas, we consider the Liouville domain  $E$  to be built from fibre  $M$  and the collection of vanishing cycles  $(V_1, \dots, V_r)$  according to Lemma 2.2. We denote by  $\Delta_i$  the Lefschetz thimble associated to  $V_i$  in the corresponding Lefschetz fibration  $\pi: E \rightarrow \mathbb{C}$ .

**Lemma 7.1.** ([1, Property 2.3]) *For a Liouville domain  $E$ , constructed from  $(M; V_1, \dots, V_m)$ ,  $SH^*(E) = 0$  if and only if  $HW^*(\Delta_i, \Delta_i) = 0$  for all  $i$ .*

**Lemma 7.2.** ([1, Property 2.5]) *Consider a Liouville domain  $E$ , constructed from  $(M; V_1, \dots, V_m)$  and let  $E'$  be the Liouville domain built from  $(M; V_2, \dots, V_m)$ . Let  $\Delta_i, \Delta'_i$  be the Lefschetz thimbles in  $E, E'$  respectively. If  $HW^*(\Delta_1, \Delta_1) = 0$  and  $HW^*(\Delta'_i, \Delta'_i) = 0$  for all  $i$ , then  $HW^*(\Delta_i, \Delta_i) = 0$  for all  $i$ .*

We also note that if  $SH^*(E; \mathbb{Z}/2) = 0$ , then  $E$  cannot contain any exact Lagrangian submanifolds [30].

Lemma 7.1 suffices to prove that Maydanskiy's exotic examples [16] have vanishing symplectic cohomology, as do the exact symplectic manifolds  $X_n^j$  considered in Section 8. We now prove that the exotic examples of Maydanskiy-Seidel, as well as their versions obtained from adding a 2-handle in the way described in Section 5 have vanishing symplectic cohomology. Take some exotic example  $X_0$  from [17], as in Figure 5.1, but without the extra rightmost critical point.

The proof in [17], as outlined in Section 5, shows that  $HW^*(\Delta_{m+1}, \Delta_{m+1}) = 0$ . We apply Lemma 7.2 in this setting, and remark that this lemma still holds if we remove the final vanishing cycle instead of the first. If we restrict to the  $A_m$  configuration of vanishing cycles  $(V_1, \dots, V_m)$  in Figure 5.1, then  $X'_0$  is just isomorphic to the standard ball. This means that if we compute  $HW^*(\Delta'_i, \Delta'_i)$ , we get zero as all the Floer groups involved in the definition of  $HW^*(\Delta'_i, \Delta'_i)$  will vanish. This suffices to prove that  $HW^*(\Delta_i, \Delta_i) = 0$  for all  $i$ , and so  $SH^*(X_0) = 0$ .

We construct the manifold  $X_2$  of Section 5 by adding a 2-handle to  $M_m$ . However, because this handle is added away from all the vanishing cycles, we can just view this as a subcritical handle added to  $X_0$ , as opposed to a critical one added to  $M_m$  since  $X_0$  is a product fibration in the region where the handle is attached. Cieliebak's result [6] says that  $SH^*(X_2) = SH^*(X_0)$  is still zero. In particular we have

**Theorem 7.3.**  *$X_2$  and  $X_0$  are both empty as exact symplectic manifolds, in the sense of containing no exact Lagrangian submanifolds.*

**Remark 7.4.** *It is sometimes possible to define symplectic cohomology with respect to some nonexact symplectic form. Ritter [24] shows that, if one performs a nonexact deformation of the exact symplectic form, then this is the same as computing the symplectic cohomology of the original structure, but with coefficients in some twisted Novikov bundle:  $SH^*(M, d\theta + \eta) = SH^*(M, d\theta; \underline{\Lambda}_{\tau\eta})$ . This has implications for the existence of exact Lagrangians and it would be interesting to compare the results of this paper with this viewpoint.*

## 8. MANY INEQUIVALENT EXOTIC SYMPLECTIC FORMS

**8.1. An invariant.** We shall now extend the ideas of Section 6 in order to prove Theorem 1.2. Suppose we have a symplectic manifold  $(E, \omega)$  which is convex at infinity and such that the map  $H^2(E; \mathbb{R}) \rightarrow H^2(\partial E; \mathbb{R})$  is zero. Then, given any cohomology class  $\eta \in H^2(E; \mathbb{R})$ , we can construct a deformation of  $E$  in the sense of Section 5 in the direction of  $\eta$ , in other words  $[\omega_t] = [\omega + t\eta] \in H^2(E; \mathbb{R})$  for some small  $\epsilon > 0$ .

Suppose in addition that  $(E, \omega)$  contains no homologically essential Lagrangian sphere. We denote by  $\Gamma_1(E, \omega)$  the set of directions  $l \in \mathbb{P}(H^2(E; \mathbb{R}))$  such that, after constructing a “small” deformation of  $(E, \omega)$  in direction  $l$ , we *still have no* homologically essential Lagrangian sphere. The Moser-type argument from Section 6 says that this set is well-defined (up to projective linear equivalence).

We can likewise consider the invariants  $\Gamma_k(E, \omega)$ , which are the set of  $k$ -planes  $P_k$  in the Grassmanian  $Gr(H^2(E; \mathbb{R}))$ , such that we get no homologically essential Lagrangian sphere *for every* direction  $l$  contained in  $P_k$ . These are again invariants up to the correct notion of linear equivalence, and so in particular, if we get a finite set of such planes, the cardinality of  $\Gamma_k(E, \omega)$  is invariant.

**8.2. The construction.** We now extend the construction of Maydanskiy [16] to exhibit, for any  $n \geq 1$ , a Liouville manifold which admits  $n + 1$  symplectic forms  $\omega_k$  all of which have no homologically essential exact Lagrangian sphere (in fact which have vanishing symplectic cohomology  $SH^*(E, \omega_k; \mathbb{Z}/2)$  and therefore no exact Lagrangian submanifolds), but such that there exists no diffeomorphism  $\phi$  of  $E$  such that  $\phi^*\omega_i = \omega_j$  for  $i \neq j$ .

Take the points  $0, 1, \dots, n + 1 \in \mathbb{C}$  and consider two paths in  $\mathbb{C}$  as in Figure 8.1. The first  $\gamma_0$  joins the extreme crosses and goes over all the others. We have some choice in the second path and denote by  $\gamma_j$  the path which goes below the points  $1, \dots, j$  and then over  $j + 1, \dots, n$ . (We include here the possibility that the second path actually goes over all central crosses and in this case just consider it to be another copy of  $\gamma_0$ .)

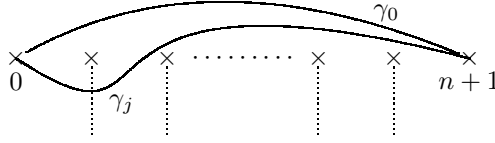


FIGURE 8.1.

With the same conventions as before, having made our choice of  $\gamma_j$ , we can associate to Figure 8.1 the 6-dimensional manifold  $(X_n^j, \omega_j)$ , which is diffeomorphically  $T^*S^3$  with  $n$  2-handles attached. It is the total space of a Lefschetz fibration whose generic fibre is the  $A_{n+1}$  Milnor fibre, which we shall denote  $M_{n+1}$ . Associated to each dotted line we get a Lagrangian 2-ball  $B_i \subset M_{n+1}$  for  $1 \leq i \leq n$ , and we denote by  $V_0$  and  $V_j$  the two matching paths associated to the paths  $\gamma_0$  and  $\gamma_j$ . If  $\gamma_j = \gamma_0$ , the 6-manifold we obtain clearly contains a Lagrangian  $S^3$ , coming from the zero-section of  $T^*S^3$ . We shall denote by  $\Delta_0, \Delta_j$  the Lefschetz thimbles associated to the two critical points of the Lefschetz fibration  $\pi: X_n^j \rightarrow \mathbb{C}$ .

$H_2(M_{n+1}; \mathbb{R}) \cong \mathbb{R}^{n+1}$  and we shall choose as our standard basis the spheres  $A_i$  given by straightline paths joining adjacent critical points  $i - 1$  and  $i$  in Figure 8.1. When included into our total space, these all determine nonzero homology classes in  $E$ , but now with the relation  $\sum A_i = 0$ . We shall therefore choose to identify  $H^2(E; \mathbb{R})$  with the  $n$ -dimensional vector space  $V = \{v \in \mathbb{R}^{n+1} : \sum v_i = 0\}$ .

Pick some vector  $\mathbf{v} = (v_1, \dots, v_n, v_{n+1}) \in V$ . By the same process as in Section 3, we can construct a deformation of the symplectic structure on  $M_{n+1}$ , by adding on 2-forms in the regions between the critical point weighted according to the components. The condition on  $\mathbf{v}$  means that the homological obstruction to the matching paths above defining matching cycles vanishes, so we can once more build the corresponding deformation of  $(X_n^j, \omega_j)$ . We are interested in what choices

of  $j$  and  $\mathbf{v}$  mean that  $(X_n^j, \omega_j)$  contains a Lagrangian sphere after the deformation corresponding to  $\mathbf{v}$ .

We first observe that, as in Section 3, we shall get a Lagrangian sphere in  $X_n^j$  when we can “lift”  $V_j$  over the critical points and onto  $V_0$ . For this to be true, we need

$$\sum_r^k v_r \neq 0 \text{ for all } k \leq j.$$

In this case we shall get a Lagrangian sphere in  $X_n^j$  once we perturb in the direction of  $\mathbf{v}$ . We shall now show that in all other cases we do not get such a sphere.

Fix some direction  $\mathbf{v} \in V$ . In what follows, we shall as before denote by  $HF_t^*$  the Floer cohomology group computed with respect to the time- $t$  deformation of  $\omega$  in the direction of  $\mathbf{v}$ . For the same reasons as already discussed, all these groups are well-defined (perhaps after rescaling  $\mathbf{v}$ ).

Suppose that there is a homologically essential Lagrangian sphere  $L \subset (X_n^j, \omega_t)$ . Then, as in Section 5, we must have  $L \cdot \Delta_j \neq 0$ , which implies that  $HW_t^*(\Delta_j, \Delta_j) \neq 0$ . This wrapped Floer group fits in an exact triangle as before.

$$(8.1) \quad \begin{array}{ccc} HF_t^*(\Delta_j, \Delta_j^\epsilon) & \longrightarrow & HF_t^*(\Delta_j, \Delta_j^{2\pi+\epsilon}) \\ & \nwarrow & \downarrow \\ & & HF_t^*(\mu(V_j), V_j). \end{array}$$

where the bottom group is calculated in the fibre  $E_z$ . Here  $\mu$  is, up to isotopy,  $\tau_{V_0} \circ \tau_{V_j}$ , so we shall need to consider the group  $HF_t^*(\tau_{V_0} V_j, V_j)$ .

The argument in this section largely follows that found in [16], from where we reproduce the following basic observation.

**Lemma 8.1.** *If we have an exact triangle of graded vector spaces*

$$\begin{array}{ccc} K & \xrightarrow{F} & L \\ & \nwarrow [1] & \downarrow \\ & & M, \end{array}$$

then  $\text{rank}(M) = \text{rank}(K) + \text{rank}(L) - 2 \text{rank}(\text{im}(F))$ .

We shall consider this lemma applied to the following triangle coming from the long exact sequence in [27].

$$\begin{array}{ccc} HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) & \longrightarrow & HF_t^*(V_j, V_j) \\ & \nwarrow & \downarrow \\ & & HF_t^*(\tau_{V_0} V_j, V_j). \end{array}$$

**Remark 8.2.** *To apply Seidel’s long exact sequence in this nonexact setting, we can no longer filter the Floer cochain groups by the symplectic action, as discussed in Section 5.3. However, we can introduce a filtration by powers of our formal*

Novikov parameter  $q$ . This will give us an appropriate  $\mathbb{Z}$ -filtration as the energy spectrum of the (unperturbed) holomorphic curves  $u$  will form a discrete set.

Consider now the Lagrangian balls  $B_i$  associated to the dotted paths in Figure 8.1 and suppose there is an  $i$  such that  $HF_t^*(V_j, B_i)$  is nonzero. Then the product

$$HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) \rightarrow HF_t^*(V_j, V_j) \cong H^*(S^2)$$

does not contain the identity in its image, because if it did, then the composite

$$HF_t^*(V_j, B_i) \otimes HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) \rightarrow HF_t^*(V_j, B_i)$$

would hit the identity despite factoring through  $HF_t^*(V_0, B_i)$  which is zero as these Lagrangians are disjoint. Here we use the fact that the product structure on Floer cohomology is associative. However, the fundamental class of  $H^2(S^2)$  is in the image, by Poincaré duality for Floer cohomology.

So, when we consider the ranks of the groups in the above triangle, we see that

**Lemma 8.3.** *If  $HF_t^*(V_j, B_i) \neq 0$  for any  $i$ , then  $\text{rank } HF_t^*(\tau_{V_0} V_j, V_j) = 4$ .*

We now consider the triangle (8.1) relating the first few terms in the system of groups computing  $HW_t^*(\Delta_j, \Delta_j)$ . Again, by computing ranks we see that, if  $\text{rank } HF_t^*(\tau_{V_0} V_j, V_j) = 4$ , then the rank of the image of the horizontal map must be zero, and therefore take 1 to 0, which in turn forces  $HW_t^*(\Delta_j, \Delta_j) = 0$ . We conclude

**Lemma 8.4.** *If  $HF_t^*(V_j, B_i) \neq 0$  for any  $i$ , then there exists no homologically essential Lagrangian sphere.*

For  $i > j$ ,  $V_j$  and  $B_i$  are disjoint so  $HF_t^*(V_j, B_i) = 0$  is automatic. For  $i \leq j$ , the criterion that  $HF_t^*(V_j, B_i)$  be nonzero corresponds to

$$\sum_r^k v_r \neq 0 \text{ for all } k \leq i$$

since, in the fibre where the paths defining  $V_j$  and  $B_i$  intersect we either get disjoint circles or instead two copies of some circle  $C$  whose self-Floer cohomology  $HF_t^*(C, C) \cong H^*(C)$  is nonzero.

**Remark 8.5.** *In particular, the above argument shows that, in the undeformed case,  $HW^*(\Delta_j, \Delta_j) = 0$ . A similar argument also shows that  $HW^*(\Delta_0, \Delta_0) = 0$ , which, by Lemma 7.1, proves that, for our undeformed exact symplectic manifolds  $SH^*(X_n^i) = 0$  for all  $i$ .*

Therefore, if we consider the  $(n-1)$ -Grassmanian invariant  $\Gamma_{n-1}(X_n^j)$ , we see that the planes for which we get no Lagrangians appearing are, in our choice of basis, precisely those  $(n-1)$ -planes defined by any one of the equations

$$\sum_r^k v_r = 0 \text{ for some } k \leq j,$$

so that  $\Gamma_{n-1}(X_n^j)$  is a set consisting of  $j$  points.

We now have, for  $1 \leq i \leq n$ , exact symplectic manifolds such  $X_n^i$  is not symplectomorphic to  $X_n^j$  for  $i \neq j$ , even though neither contains any exact Lagrangian submanifolds. Our final manifold  $(X_n^{n+1}, \omega_{n+1})$  simply comes from adding  $n$  handles to some exotic Maydanskiy-Seidel example, just as in Figure 5.1. The same argument as in Section 5 will show that  $\Gamma_{n-1}(X_n^{n+1}, \omega_{n+1}) = Gr_{n-1}(\mathbb{R}^n)$ , so  $X_n^{n+1}$  cannot be symplectomorphic to any of the  $X_n^i$  for  $i \leq n$ . This completes the proof of Theorem 1.2.

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